Advanced linear and multilinear algebra (MMA200)

Time: 2017-10-26, 14:00-18:00.

Tools: No calculator or handbook is allowed.

Questions: Jakob Hultgren, 031-7725325

Grades: Each problem gives 6 points. At most 6 bonus points from the exercise sessions will be added to the result. Grades are G (15-24 points) and VG (25-30).

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1 Define what it means for a module to be *free*. Prove or disprove that $\mathbb{Q}$ is a free module over $\mathbb{Z}$.

2 Find all abelian groups of order 360 that contain a subgroup isomorphic to $\mathbb{Z}_{36}$.

3 Let A be a complex 3×3 matrix such that $A^3 + A = 2A^2$. Determine all possibilities for the Jordan canonical form of A.

4 Formulate and prove Schur's lemma. Also give a brief explanation of how it implies orthogonality relations for irreducible characters.

5 If λ is a Young diagram, the content of a box in row i and column j is defined as $j - i$. The content $c(\lambda)$ of a diagram λ is the sum of the content of all its boxes. As an example, $c(4, 2, 1) = 3$, since

$$\begin{array}{cccccccc} & 0 & + & 1 & + & 2 & + & 3 \\ + & (-1) & + & 0 & & & & \\ + & (-2) & & & & & & = 3. \end{array}$$

Let $x = \sum_{1 \leq i < j \leq n} (ij) \in \mathbb{C}[\mathrm{S}_n]$ be the sum of all transpositions and V be the irreducible representation labelled by a Young diagram λ. Show that $\pi_V(x) = c(\lambda)\,\mathrm{Id}_V$ (here, π_V is viewed as a linear map $\mathbb{C}[\mathrm{S}_n] \to \mathrm{End}(V)$).

Advanced linear and multilinear algebra (MMA200)

2017-10-26, Solutions

1 Define what it means for a module over a ring to be *free*. Prove or disprove that $\mathbb{Q}$ is a free module over $\mathbb{Z}$.

See the course literature for the definition. Consider any two elements k/l and m/n in $\mathbb{Q}$. Then

$$lm\frac{k}{l} - kn\frac{m}{n} = 0,$$

so the two elements are linearly dependent over $\mathbb{Z}$. Thus, a basis can contain just one element k/l. But that would give $\mathbb{Q} = \mathbb{Z}k/l$, which is absurd. Thus, $\mathbb{Q}$ is not free.

2 Find all abelian groups of order 360 that contain a subgroup isomorphic to $\mathbb{Z}_{36}$.

We have $360 = 2^3 \cdot 3^2 \cdot 5$. Thus we can split such a group G as $G \simeq G_1 \times G_2 \times G_3$, where $G_1 \in \{\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2^3\}$, $G_2 \in \{\mathbb{Z}_9, \mathbb{Z}_3^2\}$ and $G_3 = \mathbb{Z}_5$. To contain a subgroup isomorphic to $\mathbb{Z}_{36} \simeq \mathbb{Z}_4 \times \mathbb{Z}_9$, we cannot choose $G_1 = \mathbb{Z}_2^3$ or $G_2 = \mathbb{Z}_3^2$, since there are then no elements of order 4 or 9. However, the other choices are allowed (in particular, $G_1 = \mathbb{Z}_8$ is allowed since $2\mathbb{Z}_8 \simeq \mathbb{Z}_4$). We conclude that there are only two such groups, $G = \mathbb{Z}_8 \times \mathbb{Z}_9$ and $G = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9$.

3 Let A be a complex 3×3 matrix such that $A^3 + A = 2A^2$. Determine all possibilities for the Jordan canonical form of A.

The given equation can be written $A(A - I)^2 = 0$. Thus, the minimal polynomial of A divides $x(x-1)^2$. This means that the only possible Jordan blocks of A are $[0]$, $[1]$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Up to reordering these blocks, we have six possibilities:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4 Formulate and prove Schur's lemma. Also give a brief explanation of how it implies orthogonality relations for irreducible characters.

See the course literature.

5 If λ is a Young diagram, the content of a box in row i and column j is defined as $j - i$. The content $c(\lambda)$ of a diagram λ is the sum of the content of all its boxes. As an example, $c(4, 2, 1) = 3$, since

$$\begin{array}{lllllll} & 0 & + & 1 & + & 2 & + \; 3 \\ + & (-1) & + & 0 & & & \\ + & (-2) & & & & & = 3. \end{array}$$

Let $x = \sum_{1 \le i < j \le n} (ij) \in \mathbb{C}[\mathrm{S}_n]$ be the sum of all transpositions and V be the irreducible representation labelled by a Young diagram λ. Show that $\pi_V(x) = c(\lambda) \, \mathrm{Id}_V$ (here, π_V is viewed as a linear map $\mathbb{C}[\mathrm{S}_n] \to \mathrm{End}(V)$).

As we discuss in §5.9 of the lecture notes, the element $\sum_{g \in C} g$ is central in the group algebra for any conjugacy class C of a finite group. In particular, x is central. Thus, with $a_\lambda = \sum_{p \in P} p$ and $b_\lambda = \sum_{q \in Q} \mathrm{sgn}(q) q$ as defined in the lecture notes, we can write

$$\pi_V(x) g a_\lambda b_\lambda = g \sum_{1 \le i < j \le n} a_\lambda (ij) b_\lambda,$$

where the elements $ga_\lambda b_\lambda$ generate V. Note that if $(ij) \in P$ then $a_\lambda(ij) = a_\lambda$, and if $(ij) \in Q$ then $(ij)b_\lambda = -b_\lambda$. We claim that if neither of these conditions hold, then $a_\lambda(ij)b_\lambda = 0$. In this case, i and j are neither in the same row nor in the same column. Possibly after interchanging the names of i and j, we can then find k in the row of i and the column of j, so that $(ik) \in P$ and $(jk) \in Q$. Moreover, $(ik)(ij) = (ij)(jk) = (ijk)$, so $a_\lambda(ij)b_\lambda = a_\lambda(ik)(ij)b_\lambda = a_\lambda(ij)(jk)b_\lambda = -a_\lambda(ij)b_\lambda$, which gives $a_\lambda(ij)b_\lambda = 0$. (This is a special case of the proof of Lemma 6.2.5 in the lecture notes). We conclude that $\pi_V(x) = c\,\mathrm{Id}_V$, where c is the number of transpositions in P minus the number of transpositions in Q. To count the transpositions in P, note that a box in column j is the right-most member of $j-1$ such transpositions, so the total number is $\sum_{i,j}(j-1)$, where the sum runs over all boxes. Similarly, there are $\sum_{i,j}(i-1)$ transpositions in Q. Combining these expressions give $c = \sum_{i,j}(j-i) = c(\lambda)$.