Computation of a tensor product

One of the exercises last time was to compute the real vector space

$$V = \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}^2 \times \mathbb{Z}_3).$$

Unfortunately I was not sufficiently prepared and had underestimated the difficulty of this exercise. Here is a solution.

We first note that

$$t \otimes (k, l, m) = \frac{t}{3} \otimes (3k, 3l, 3m) = \frac{t}{3} \otimes (3k, 3l, 0) = t \otimes (k, l, 0),$$

where $t \in \mathbb{R}$, k, $l \in \mathbb{Z}$, $m \in \mathbb{Z}_3$. It follows that

$$V \simeq \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}^2.$$

We now apply the following lemma.

Lemma 1. *When R is a commutative ring and U is any R-module, then*

$$U \otimes_R R^n \simeq U^n \tag{1}$$

as R-modules.

Proof. We will show that U^n has the appropriate universal property. That is, we display an R-bilinear map $\pi : U \times R^n \to U^n$ such that, for any R-module W and R-bilinear map $f : U \times R^n \to W$, there exists a unique R-linear map $g : U^n \to W$ with $f = g \circ \pi$. We claim that

$$\pi(u, r_1, \dots, r_n) = (r_1 u, \dots, r_n u)$$

does the job. The equation $f = g \circ \pi$ then gives

$$f(u, r_1, \dots, r_n) = g(r_1 u, \dots, r_n u).$$

Choosing $r_j = \delta_{jk}$, this gives

$$f(u, e_k) = g(0, \dots, u, \dots, 0),$$

where $(e_k)_{k=1}^n$ are the standard basis vectors of R^n and the u on the right appears in position k. It follows that

$$\begin{aligned} g(u_1, \dots, u_n) &= g(u_1, \dots, 0) + \dots + g(0, \dots, u_n) \\ &= f(u_1, e_1) + \dots + f(u_n, e_n). \end{aligned} \tag{2}$$

This shows that if g exists it is unique and given by (2). Conversely, defining g by (2) we have

$$\begin{aligned}(g \circ \pi)(u, r_1, \ldots, r_n) &= g(r_1 u, \ldots, r_n u) = f(r_1 u, e_1) + \cdots + f(r_n u, e_n) \\ &= f(u, r_1 e_1) + \cdots + f(u, r_n e_n) = f(u, r_1 e_1 + \cdots + r_n e_n) \\ &= f(u, r_1, \ldots, r_n).\end{aligned}$$

Thus, we indeed have $f = g \circ \pi$. By Prop. 3.6.2 in the lecture notes, it follows that

$$\phi(u \otimes (r_1, \ldots, r_n)) = \pi(u, r_1, \ldots, r_n) = (r_1 u, \ldots, r_n u) \tag{3}$$

defines an isomorphism $\phi : U \otimes_R R^n \to U^n$. $\square$

We are not quite done yet! The lemma just shows that $V \simeq \mathbb{R}^2$ as $\mathbb{Z}$-modules. We must show that the isomorphism respects multiplication by real scalars. By (3), the isomorphism $\phi : V \to \mathbb{R}^2$ is determined by $\phi(t \otimes (k, l, m)) = (tk, tl)$. The scalar multiplication on V is defined by $s(t \otimes (k, l, m)) = st \otimes (k, l, m)$. The image of this element under ϕ is $(stk, stl) = s(tk, tl)$. This shows that the $\mathbb{Z}$-linear isomorphism ϕ is also $\mathbb{R}$-linear.