Multilinear algebra 2017. Check-list for theory.

This is a list of the main definitions and theorems that are included in the course. As for the proofs, you should use your own judgement and study the proofs that seem most important/interesting/instructive.

I marked a few more advanced/peripheral topics with a $\circ$. If your aim is just to pass the exam, you don't need to spend a lot of effort on these parts. (To be clear, there will be at most one question on these topics combined, and it will be considered as an advanced question.)

Fundamental notions of group and ring theory (prerequisites)

- Basic notions of group theory: Group, abelian/commutative group, group homomorphism, subgroup, coset, normal subgroup, quotient group, direct product of groups.
- Examples of groups: Cyclic groups ($\mathbb{Z}_n$ and $\mathbb{Z}$), symmetric group, general linear group.
- Fundamental theorem of group homomorphisms: $G/\operatorname{Ker}(\phi) \simeq \operatorname{Im}(\phi)$.
- What is the cycle decomposition of a permutation? How does one compute with cycle decompositions (group multiplication, conjugation). What are the conjugacy classes of S_n? How many elements are there in each conjugacy class?
- What is the sign of a permutation, and how is it related to the determinant?
- Basic notions of ring theory: Ring, ring homomorphism, subring, ideal, left ideal, quotient, direct product.
- Examples of rings: Number fields, residue class rings, polynomials, matrices.
- Special types of rings: Commutative ring, Integral domain, UFD, PID, Field. To which types do e.g. $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{Z}[x, y]$, $\mathbb{R}$, $\mathbb{R}[x]$, $\mathbb{R}[x, y]$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, belong?
- Types of ideals: Finitely generated, principal.
- Fundamental theorem of ring homomorphisms.

More advanced notions of group and ring theory

- Group algebra of a group.
- Irreducible element and prime element in an integral domain.
- Relation between unique factorization and irreducible elements being prime.
- Every PID is Noetherian.
- Every PID is an UFD.

Modules

- Basic notions of module theory: Module, Module homomorphism, Submodule, Quotient module, Direct sum, Dual module, Free module, Cyclic module, Finitely generated module, Basis for a module, Annihilator of an element or a subset, Torsion module, Torsion-free module.
- Why is a $\mathbb{Z}$-module the same thing as an abelian group?
- Fundamental theorem of module homomorphisms.
- Relation between cyclic modules and quotients of rings by left ideals.
- If two free modules have bases with the same cardinality, are they then isomorphic? If two free modules are isomorphic, do any two bases for them necessarily have the same cardinality? (Answer: Yes, No.)
- Prove that for a finitely generated free module over a commutative ring, any two bases have the same cardinality.
- Different characterizations of direct sums of modules.
- Exact sequence, short exact sequence, split exact sequence.
- Relation between split exact sequences and direct sums.
- Prove that if M/L is free, then L is a direct summand of M.

Tensor products

- Define the tensor products of modules over a commutative ring. Give both a construction and a characterization by a universal property.
- For vector spaces with given bases, write down a basis for the tensor product.
- Basic properties of tensor products: Commutativity, associativity, distributivity with respect to direct sums.
- How can tensor products be used to "extend scalars"?
- What is the symmetric and exterior product? Describe how they can be viewed both as subspaces and as quotients of the tensor product.
- If V is a vector space, how can we obtain bases for $V^{\odot n}$ and $V^{\wedge n}$?
- How do we define the tensor algebra, the symmetric algebra and the Grassmann algebra? Does $u \wedge v = -v \wedge u$ hold in the Grassmann algebra?

- What is the relation between subspaces of a vector space and pure tensors in the Grassmann algebra?

○ How can we use the Grassmann algebra to describe the space of all k-dimensional subspaces of a vector space? How does this imply the Plücker minor relations?

Modules over a PID

- Classification of finitely generated modules over a PID; classification of finitely generated abelian groups; classification of finite abelian groups.
- What do the numbers $\dim_{R/pR}(p^k M/p^{k+1}M)$ (with $p \in R$ a prime element) say about M?
- What is the primary decomposition and the invariant factor decomposition? How to pass between them?
- How can we use the general theory of modules over a PID to study matrices?
- What is the primary canonical form and rational canonical form of a matrix? How can they be obtained in practice? Do they change if we change the base field?
- What is the Jordan canonical form of a complex matrix? How can it be obtained in practice?
- What is the minimal polynomial of a matrix? How can we find the minimal polynomial and characteristic polynomial given the rational canonical form or the Jordan canonical form?
- Cayley–Hamilton theorem.

Group representations

- Basic notions: Group representation, Intertwining map, Equivalence of group representations, Subrepresentation, Irreducible representation, Class function, Character, Trivial representation, Regular representation.
- If V and W are representations of G, how do we define a representation of G on V^*, $V \oplus W$, $V \otimes W$ and $\mathrm{Hom}(V, W)$? What are the characters of those representations?
- Maschke's theorem.
- Schur's lemma.
- Why do representations carry an invariant inner product?
- How does the regular representation split into irreps?
- How is the order of the group related to the dimension of the irreps?

- Why does a finite group have as many conjugacy classes as irreps?
- Why is a representation determined by its character?
- Orthogonality of characters (there are two such relations, which are dual to each other).
- Given the character table, how do we find: The order of the group, the order of the conjugacy classes, the normal subgroups, the center of the group, the commutator subgroup.
- Describe the representations of a finite abelian group.
- Decomposition of the group algebra as a representation of $G \times G$ and as an associative algebra. What are the interwining maps from $\mathrm{End}(V)$ ($V \in \mathrm{Irr}(G)$) into the group algebra (Fourier inversion)?
- Peter–Weyl theorem for finite groups.

◦ Frobenius divisibility theorem.

- Every irrep is a left ideal $\mathbb{C}[G]e$, with $e^2 = e$. Given an idempotent e, how can we see whether $\mathbb{C}[G]e$ is irreducible? If $\mathbb{C}[G]e$ and $\mathbb{C}[G]f$ are irreducible, how can we see whether $\mathbb{C}[G]e \simeq \mathbb{C}[G]f$?

Representations of the symmetric group

- How can one construct all irreps of S_n?

◦ How can one compute the character table of S_n? What is the dimension of the irreps?

- How does $V^{\otimes n}$ decompose into irreps under the natural action of the groups S_n, $\mathrm{GL}(V)$ and $\mathrm{S}_n \times \mathrm{GL}(V)$?

◦ What are the character of those irreps of $\mathrm{GL}(V)$ that appear in $V^{\otimes n}$? What are their dimensions?

Compact groups

- What is the Haar measure on a compact group? In particular, what is the Haar measure on $\mathrm{U}(1)$ and $\mathrm{SU}(2)$?

◦ Which basic facts on representations of finite groups can be extended to compact groups? How?

- Describe all irreducible represeentations of $\mathrm{SU}(2)$. What are their characters?
- Explain how the matrix elements for $\mathrm{SU}(2)$ can be written as hypergeometric sums.

◦ What are Krawtchouk and Jacobi polynomials? Explain how their orthogonality relations can be obtained using representation theory.