Assignment 2: Number Theory

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Discriminants (4 pt) Let $\alpha_i \in \mathbb{C}$ be the roots of the irreducible polynomial $x^3 + 4x^2 + x - 1$, and set $L = \mathbb{Q}(\alpha_1) = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Compute the discriminant of $\mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_1^2$ using the two formulas

$$d(1, \alpha_1, \alpha_1^2) = \det\Big(\begin{pmatrix} \sigma_1 1 & \sigma_1 \alpha_1 & \sigma_1 \alpha_1^2 \\ \sigma_2 1 & \sigma_2 \alpha_1 & \sigma_2 \alpha_1^2 \\ \sigma_3 1 & \sigma_3 \alpha_1 & \sigma_3 \alpha_1^2 \end{pmatrix} \Big)^2, \quad \mathrm{Hom}_{\mathbb{Q}}(L, L) = \big\{ \sigma_1, \sigma_2, \sigma_3 \big\}$$

and

$$d(1, \alpha_1, \alpha_1^2) = \det\Big(\begin{pmatrix} \mathrm{Tr}_{L/\mathbb{Q}}(1 1) & \mathrm{Tr}_{L/\mathbb{Q}}(1 \alpha_1) & \mathrm{Tr}_{L/\mathbb{Q}}(1 \alpha_1^2) \\ \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1 1) & \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1 \alpha_1) & \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1 \alpha_1^2) \\ \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1^2 1) & \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1^2 \alpha_1) & \mathrm{Tr}_{L/\mathbb{Q}}(\alpha_1^2 \alpha_1^2) \end{pmatrix} \Big).$$

For the first computation, you may use the expressions $\alpha_2 = \alpha_1^2 + 3\alpha_1 - 2$ and $\alpha_3 = -\alpha_1^2 - 4\alpha_1 - 2$ that were derived in the first assignment. For the second computation do not use the α_2 or α_3.

Rings of integers: The pure cubic case (3 pt) Show that 1, $\sqrt[3]{2}$, $\sqrt[3]{2}^2$ is an integral basis of $\mathbb{Q}(\sqrt[3]{2})$.

Rings of integers: The general cubic case (3 pt) Show that 1, α, $\frac{1}{2}(\alpha + \alpha^2)$ is an integral basis of $\mathbb{Q}(\alpha)$, where α is a root of the irreducible polynomial $x^3 - x - 4$.