Assignment 3: Number Theory

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Factorization into irreducible elements (1 pt) Find all factorizations of 21 into irreducible elements of $\mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-13})$, and determine how many essentially different (i.e. up to reordering and multiplication by units) factorizations there are.

Factorization into prime ideals (1 pt) Factor the ideal $(21) \subset \mathcal{O}_K$ for $K = \mathbb{Q}(\sqrt{-13})$ into prime ideals.

Chinese remainder theorem (2 pt) Decompose the ring $\mathcal{O}_K/(21)$ for $K = \mathbb{Q}(\sqrt{-13})$ into a direct sum of fields, and given an explicit isomorphism.

Multimodular linear algebra (6 pt) Let $M \in \mathrm{Mat}_n(\mathcal{O}_K)$ be an $n \times n$ matrix with entries in $\mathcal{O}_K$ for an imaginary quadratic number field K. Describe an algorithm to compute the row Echelon normal form of M over K from the row Echelon normal forms of $M \pmod{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$.

Applying the algorithm, compute the row Echelon normal form of

$$\begin{pmatrix} 14a & 5a-30 & 5a+65 \\ -16a & -5a+28 & -3a-72 \\ -40a+4 & -20a+98 & -22a-202 \end{pmatrix}$$

over $K = \mathbb{Q}(\sqrt{-5})$ with $a = \sqrt{-5}$, and compare this with direct application of the Gaussian algorithm.

Noetherian and artinian rings (5 pt) Show that in a noetherian ring R in which every prime ideal is maximal, every descending chain of ideals becomes stationary.

Representatives of class group elements (5 pt) Let $\mathfrak{m}$ be a nonzero integral ideal of the Dedekind domain $\mathcal{O}_K$ for K a number field. Show that every ideal class in Cl_K, there exists an integral ideal prime to $\mathfrak{m}$.