MMA320 Introduction to Algebraic Geometry

Exercises and Home work V 6

Exercises

1. Dolgachev 5.5, 5.6, 7.2, 6.2

2. Hand out on Bezout: exercises 3 – 7

Home work, to be handed in wednesday 2010-02-17.

1. a) Prove Gauß' Lemma: if A is a UFD and F, $G \in A[X]$ are polynomials with coefficients in A, then a prime element of A that is a common factor of the coefficients of the product FG is a common factor of the coefficients of F or G.
b) Let A be a UFD. Prove that $A[X]$ is a UFD. For this you need to compare factorisations in $A[X]$ with factorisations in $Q(A)[x]$, where $Q(A)$ is the field of fractions of A, using Gauß' lemma to clear denominators.

2. Let $f(x,y)$ be the affine equation of a real or complex plane curve, and $p=(a,b)$ a point on it; suppose that $\frac{\partial f}{\partial y}(p) \neq 0$, so by the implicit function theorem $y=\varphi(x)$ in a neighbourhood of p. Prove that p is an inflection point (in the sense that $\varphi''(a)=0$) if and only if:

$$\begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{vmatrix} = 0.$$

(Hint: differentiate $f(x,\varphi(x)) \equiv 0$ twice. Compute also the determinant.)
Use Euler's formula and $f(a,b)=0$ to translate this condition into the condition on the vanishing of the Hessian of the associated homogeneous polynomial $F(X,Y,Z)$.

3. a) Give an example (in char p) of an irreducible curve in the projective plane with identically vanishing Hessian, but where not all points are inflection points.

b) Let C_1, $C_2 \subset \mathbb{P}^2(K)$, K algebraically closed of characteristic zero, be cubic curves with homogeneous equations

$$\begin{aligned} C_1\colon &\quad (X+Z)^3+3Y^3-Z^3=0, \\ C_2\colon &\quad X^3+X^2(Y+Z)+XZ^2+Y^3=0. \end{aligned}$$

Compute, using the resultant of these polynomials, the intersection points of C_1 and C_2 with their multiplicities.