

MMA320 Introduction to Algebraic Geometry
Exercises for Chapter 2

41. What points in \mathbb{P}^2 do not belong to two of the three sets $\mathbb{A}_0^2, \mathbb{A}_1^2, \mathbb{A}_2^2$?
42. a) Describe the curve $C_1: 2X + Y^2 = 1$ in the other two standard coordinate charts on $\mathbb{P}^2(\mathbb{C})$. Hint: first homogenise the equation with the coordinate Z .
b) Let C_2 be defined by the equation $Y = X^3$ in the affine chart $Z = 1$. What does C_2 look like at infinity? Give its equation and draw its real part.
c) Find all the points of \mathbb{P}^2 which lie in both curves C_1 and C_2 .
43. a) Let $C_1: y = x^2 + 1$ and $C_2: y = 0$. What is $C_1 \cap C_2$ in $\mathbb{A}^2(\mathbb{R})$ respectively $\mathbb{A}^2(\mathbb{C})$? Does anything change if we make the equations homogeneous and think of the curves as lying in \mathbb{P}^2 . Explain this in terms of ‘asymptotic directions’.
b) Let C_k be the circle $x^2 + y^2 = k^2$ in $\mathbb{A}^2(\mathbb{R})$. Show that $C_1 \cap C_2 = \emptyset$. What happens if we replace \mathbb{R} with \mathbb{C} ? What about $\mathbb{P}^2(\mathbb{C})$?
44. Two conics in $\mathbb{P}^2(\mathbb{C})$ have four intersection points (counted with multiplicity). Give an example of two conics in $\mathbb{P}^2(\mathbb{R})$ which only intersect in 2 points.
Find the extra intersection points when you use the same equations to define conics in $\mathbb{P}_{\mathbb{C}}^2$. [This may or may not be difficult depending on your choice of equation.] Why can you deduce that these intersection points are distinct (even without calculating them)?
45. Let $k = \mathbb{Z}/(2)$ be the field with two elements. How many points has $\mathbb{P}^2(k)$? How many lines pass through $P = (1 : 0 : 0)$? How many points lie on each of these lines? Draw all points and lines. Hint: choose $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ as vertices in a triangle and $(1 : 1 : 1)$ as interior point.
46. **Duality.** Let $\{e_0, e_1, e_2\}$ be a basis of $V \cong \mathbb{R}^3$ and let $(X_0 : X_1 : X_2)$ be corresponding homogeneous coordinates on $\mathbb{P}(V) \cong \mathbb{P}^2(\mathbb{R})$. Let $\{e_0^*, e_1^*, e_2^*\}$ be the dual basis in V^* , and $(U_0 : U_1 : U_2)$ corresponding homogeneous coordinates on $\mathbb{P}(V^*) \cong \mathbb{P}^2(\mathbb{R})$. Let $P = (a_0 : a_1 : a_2)$ be a point in $\mathbb{P}(V)$. Describe the pencil of all lines in $\mathbb{P}(V)$ through P in the coordinates $(U_0 : U_1 : U_2)$.
47. Let l_1 and l_2 be two disjoint lines in \mathbb{P}^3 , and let $P \in \mathbb{P}^3 \setminus (l_1 \cup l_2)$ be a point. Show that there is a unique line $l \subset \mathbb{P}^3$ through P , intersecting l_1 and l_2 and P .
48. Let P_0, P_1, P_2 (resp. Q_0, Q_1, Q_2) be three points in \mathbb{P}^2 not lying on a line. Show that there is a projective change of coordinates $T: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that $T(P_i) = Q_i$, $i = 0, 1, 2$. Extend this to n points in \mathbb{P}^n , not lying on a hyperplane.
49. Let l_0, l_1, l_2 (resp. m_0, m_1, m_2) be lines in \mathbb{P}^2 that do not all pass through a point. Show that there is a projective change of coordinates $T: \mathbb{P}^2 \rightarrow \mathbb{P}^2$ such that $T(l_i) = m_i$, $i = 0, 1, 2$. (Hint: Let $P_i = L_j \cap L_k$, $Q_i = m_j \cap m_k$.)
50. Let $f \in k[X_0, \dots, X_n]$. Define the (formal) derivative $\frac{\partial f}{\partial X_i} \in k[X_0, \dots, X_n]$, for any field k . (Hint: product rule).

51. Let $f \in k[X_0, \dots, X_n]$ be a homogeneous polynomial of degree m . Show Euler's formula

$$\sum_{i=0}^n X_i \frac{\partial f}{\partial X_i} = mf. \quad (*)$$

When does the converse hold: for which fields does $(*)$ imply that $f \neq 0$ is homogeneous of degree m .

52. Let $I \subset k[X_0, \dots, X_n]$ be a homogeneous ideal. Show that I is prime if and only if for every two homogeneous elements $f, g \in I$ we have that $fg \in I$ implies $f \in I$ or $g \in I$.
53. The sum, product, intersection and radical of homogeneous ideals are also homogeneous ideals.
54. Show that an ideal $I \subset k[X_1, \dots, X_n]$ is prime if and only if its homogenisation $\bar{I} \subset k[X_0, \dots, X_n]$ is prime.
55. Find I^{sat} , where $I = (X^2, XY) \subset k[X, Y]$.
56. Let $k = \mathbb{Z}/(2)$ be the field with two elements. Determine for $V(x^2 + yz) \subset \mathbb{A}^3(\mathbb{Z}/(2))$ the ideal $I(V(x^2 + yz)) \subset \mathbb{Z}/(2)[x, y, z]$. Now consider the same equation in the projective plane: $V(X^2 + YZ) \subset \mathbb{P}^2$ and find the homogeneous ideal $J(V(X^2 + YZ)) \subset \mathbb{Z}/(2)[X, Y, Z]$.

57. Let $C \subset \mathbb{P}^3$ be the rational normal curve of degree 3, given by the parametrization

$$\mathbb{P}^1 \rightarrow \mathbb{P}^3, \quad (S : T) \mapsto (X : Y : Z : W) = (S^3 : S^2T : ST^2 : T^3).$$

Let $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $Z = 0$. Let π be the projection from P to H , i.e. the map associating to a point Q of C the intersection point of H with the unique line through P and Q .

- a) Show that π is a morphism.
- b) Determine the equation of the curve $\pi(C)$ in $H \cong \mathbb{P}^2$.
- c) Is $\pi: C \rightarrow \pi(C)$ an isomorphism onto its image?
58. Show that the curve $C = V(X^3 - ZY^2) \subset \mathbb{P}^2$, defined over an algebraically closed field k , is birational to \mathbb{P}^1 . Consider the affine chart $U_0 = \{Z \neq 0\}$. Are the coordinate rings of the affine curve $C \cap U_0$ and \mathbb{A}^1 isomorphic? Does there exist an affine chart U_1 such that $C \cap U_1$ has coordinate ring, isomorphic to $k[t]$?
59. Let L be the vector space of homogeneous polynomials of degree 2 in $k[X : Y : Z]$ and let $\mathbb{P}(L)$ be the linear system of conics.
- a) Show that the space of irreducible conics in \mathbb{P}^2 is an open subset $U \subset \mathbb{P}(L)$. What geometric objects can be associated to the points in $\mathbb{P}(L) \setminus U$?
- b) Show that it is a linear condition in $\mathbb{P}(L)$ for the conics to pass through a given point in \mathbb{P}^2 . More precisely, show that there is a linear subspace $L_P \subset \mathbb{P}(L)$ such that the conics passing through P are exactly those in L_P .
- c) Prove that there is a unique irreducible conic through any five given points, such that no three of them lie on a line. What happens if three of them do lie on a line?
60. Show that the plane curves $C_1 = V(ZY^2 - X^3 + XZ^2)$ and $C_2 = V(X^3Z - Y^2Z^2 + X^2Y^2)$ are birational (hint: standard Cremona transformation). Describe occurring singularities.

61. Let $H_d \subset \mathbb{P}^n$ be a hypersurface of degree d . Show that the complement $\mathbb{P}^n \setminus H_d$ is an affine variety.
62. Let $f(X_0, \dots, X_n) = g(X_1, \dots, X_n) + X_0 h(X_1, \dots, X_n)$, where g is a homogeneous polynomial of degree d and h is a homogeneous polynomial of degree $d-1$. Assuming that f is irreducible, prove that the variety $V(f)$ is rational.
63. Show that every isomorphism $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a projective transformation.
64. Is the union (resp. the intersection) of quasi-projective algebraic sets a quasiprojective algebraic set?
65. Let V be a projective variety over an algebraically closed field k . Show that $\mathcal{O}(V) = k$, that is, every rational function, regular on the whole of V , is constant. Hints: show that on each standard open affine set V_i an $r \in \mathcal{O}(V)$ has the form $r = f_i/X_i^{N_i}$ with $f_i \in S_{N_i}(V)$ homogeneous of degree N_i . Show that for N sufficiently large multiplication with r is an endomorphism of the vector space $S_N(V)$. Use the Theorem of Cayley–Hamilton to conclude that r is a root of a polynomial with coefficients in k ; alternatively, consider an eigen vector and show that r equals the eigen value.
66. Show that every regular map from a projective variety to an affine variety maps to a point.
67. Let $C = V(f) \subset \mathbb{A}^2(\mathbb{C})$ and consider the blow up $\sigma: \text{Bl}_0\mathbb{A}^2 \rightarrow \mathbb{A}^2$. Put $\tilde{f} = f \circ \sigma$. If $0 \in V(f)$, then the exceptional curve E is an irreducible component of $V(\tilde{f})$, and the closure of $V(\tilde{f}) \setminus E$ is the strict transform \bar{C} of C . Compute an equation for \bar{C} for the following curves:
- $x^2 - y^2$,
 - $x^2 - y^3$,
 - $x^2 - y^n, n \geq 4$,
 - $x^4 - y^4$.