MMA320 Introduction to Algebraic Geometry

Exercises for Chapters 5 and 6

111. Show that if k is a field of characteristic $p > 0$, and $f \in k[X_1, \ldots, X_n]$, then $\frac{\partial f}{\partial X_i} = 0$ if and only if $f \in k[X_1, \ldots, X_i^p, \ldots, X_n]$. Conclude that all the $\frac{\partial f}{\partial X_i}$ are zero if and only if $f \in k[X_1^p, \ldots, X_n^p]$.

112. Let $V \subset \mathbb{P}^n$ be a projective variety of dimension d. Show that there exists disjoint linear subspaces L_d and L'_{n-d-1} of dimension d and $n-d-1$, such that projection from L'_{n-d-1} maps V onto l_d and over every point of L_d lie only finitely many points of V.

113. Let $V \subset \mathbb{P}^n$ be a smooth variety. Show that the image of V under the projection from a point $P \in \mathbb{P}^n$ outside V is smooth if and only if for all lines l through P the intersection multiplicity of V and l is at most 1, that is no line intersects V in two points, or is a tangent.

114. Show that a smooth variety $V \subset \mathbb{P}^n$ of dimension d can be projected isomorphically onto a smooth variety in $\mathbb{P}^{2d+1}$.

115. Prove that the intersection of a hypersurface $V(F) \subset \mathbb{A}^n$ (not a hyperplane) with the tangent hyperplane $T(V)_p$ in a nonsingular point $p \in V$ is singular at p. (Here one defines the intersection by the ideal (F, L), where L is the linear equation for the tangent hyperplane.)
Show that this need not be true if one defines the intersection with reduced structure, i.e., by the radical of the ideal (F, L).

116. Find the singular points of the surface $Z^2 = X^2Y$ in $\mathbb{A}^3$. Make a sketch of the real points of this surface. Describe also its projective closure.

117. Find the singular points of the Steiner quartic $V(X_2X_3 + X_1X_3 + X_1X_2 - X_0X_1X_2X_3)$ in $\mathbb{P}^3$.

118. Determine the singular locus of $V(I) \subset \mathbb{A}^7$, where I is the ideal

$$(A_1D_1 - SA_2D_2,\ B_1D_1 - SB_2D_2,\ A_1D_2 + A_2D_1,\ B_1D_2 + B_2D_1).$$

Find the irreducible components.

119. Find an ideal of $k[X_1, \ldots, X_n]$ which admits minimal sets of generators differing in their number of elements; here minimal means that no proper subset generates.

120. Suppose a hypersurface $V = (f)$ of degree $d > 1$ in $\mathbb{P}^n$ contains a linear subspace of dimension $r \geq n/2$. Show that V is singular.

121. Prove that there is at least one line through a singular of a cubic surface (and 'in general' 6). Let $X \subset \mathbb{P}^4$ be a non-singular cubic threefold. Show that through every point there is at least one line.

122. Show that the surface $V(X^n + Y^n + Z^n + T^n) \subset \mathbb{P}^3$ contains exactly $3n^2$ lines (char k and n coprime).

123. Find all 27 lines on Clebsch' diagonal surface:

$$\begin{aligned} X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 &= 0 \\ X_0 + X_1 + X_2 + X_3 + X_4 &= 0 \end{aligned}$$

All lines are real, 15 are easy to see as intersections with coordinate hyperplanes.

124. Find the singular points of the cubic surface

$$X_0X_1X_2 + X_0X_1X_3 + X_0X_2X_3 + X_1X_2X_3 = 0 \; .$$

Determine all lines on the surface.

125. Find the singular point of the cubic surface

$$ZX^2 - Y^3 + TZ^2 = 0 \; .$$

Determine all lines on the surface.

126. Determine the number of lines on the singular cubic surface

$$XYZ - T^3 = 0 \; .$$

127. Let $P_1, \ldots, P_6$ be 6 points on a nondegenerate conic $C0(f) \subset \mathbb{P}^2$. The linear system $\mathbb{P}(S_3(P_1, \ldots, P_6))$ of cubics through these six points gives rise to a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$. Compute the equation of the image.

128. Let π be the skew projection of a smooth cubic surface from two skew lines l and m on a plane through the transversal l_1. Show that π is a regular map and determine which lines are blown down.

129. Let S be the cubic surface, which is the image of the rational map determined by the linear system of cubics in the plane through 6 points, no three on a line, not all on a conic. Describe the 27 lines in terms of points and curves in the plane.