MMA320 Introduction to Algebraic Geometry
Exercises for Chapter 1

1.1. Is A a noetherian ring if $A[X]$ is noetherian?

1.2. Which of the following rings are Noetherian:
i) the polynomial ring $k[X_1, X_2, \ldots, X_n, \ldots]$ in infinitely many variables X_1, X_2, $\ldots$, X_n, $\ldots$.
ii) the ring of all power series $a_0+a_1z+a_2z^2+\ldots$ with positive radius of convergence, with all $a_i \in \mathbb{C}$.

1.3. a) Let A be a Noetherian ring and I an ideal. Show that the quotient ring $B = A/I$ is Noetherian.
b) Let A be a Noetherian integral domain and K its field of fractions; let $0 \notin S \subset A$ be a subset and set

$$B = A_S = \{a/b \in K \mid a \in A, \text{ and } b = 1 \text{ or a product of elements of } S\} .$$

Show that B is Noetherian.

1.4. Find $I(V(J))$ for the following ideals in $\mathbb{R}[X, Y]$ and discuss the geometric meaning in each case:
a) $J = (X - a, Y - b)$, $a, b \in \mathbb{R}$ b) $J = (X^2 + Y^2)$ c) $J = (X^2 + Y^2 + 1)$
d) $J = (X^2 + Y^2 - 1, 2X^2 + Y^2 - 2)$ e) $J = (X^2, XY)$ f) $J = (X^2Y, XY^2)$

1.5. Find an ideal I such that $V(I)$ consists of the union of the (X_1, X_2)-plane and the (X_3, X_4)-plane in $\mathbb{A}^4(k)$, for any field k. In the case $k = \mathbb{R}$ find one polynomial with the same two planes as zero set.

1.6. If k is a finite field, show that every subset of $\mathbb{A}^n(k)$ is algebraic, in fact a hypersurface (i.e, given by one equation).

1.7. Give an example of a countable collection of algebraic sets whose union is not algebraic.

1.8. Suppose C is an affine plane curve, and L is a line in $\mathbb{A}^2(k)$, L not a component of C. Suppose that $C = V(f)$ for $f \in k[X, Y]$ a polynomial of degree n. Show that $L \cap C$ is a finite set of no more than n points. (Hint: Suppose $L = V(Y - (aX + b))$, and consider $f(X, aX + b) \in k[X]$.)

1.9. a) Show that an algebraic set in $\mathbb{A}^1(k)$ is finite or the whole of $\mathbb{A}^1(k)$.
b) Let $f, g \in k[X, Y]$ be polynomials without common factor. Show that $V(f, g)$ is finite.
c) Show that every algebraic set $V \subset \mathbb{A}^2(k)$ is a finite union of points and curves.

1.10. a) Let k be an infinite field and $f \in k[X_1, \dots, X_n]$ a non-constant polynomial. Show that $V(f) \neq \mathbb{A}^n(k)$.
b) Suppose now that k is algebraically closed. Let f as in b) have leading term of the form $a_m(X_1, \dots, X_{n-1})X_n^m$. Show that, if $a_m \neq 0$, there is a finite non-empty set of points in $V(f)$ for every value of $(X_1, \dots, X_{n-1})$. Conclude that $V(f)$ is infinite for $n \geq 2$.

1.11. What is the closure of the subset $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ in the Zariski topology?

1.12. Let $J = (XY, XZ, YZ) \subset k[X, Y, Z]$. Find $V(J) \subset \mathbb{A}^3(k)$. Is it irreducible?. Prove that J cannot be generated by two elements. Now let $J' = (XY, (X - Y)Z)$. Find $V(J')$ and compute $\sqrt{J'}$.

1.13. Let $J = (X^2 + Y^2 - 1, Y - 1)$. Find an $f \in I(V(J)) \setminus J$.

1.14. Let $J = (X^2 + Y^2 + Z^2, XY + XZ + YZ)$. Determine $V(J)$ and $\sqrt{J}$.

1.15. Let $V \subset \mathbb{A}^n(k)$, $W \subset \mathbb{A}^m(k)$ be algebraic sets. Show that

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \mid (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

is an algebraic set in $\mathbb{A}^{n+m}(k)$. It is called the product of V and W.
Give an example to show that the Zariski topology on $V \times W$ is not the product topology of the Zariski topologies on V and W.

1.16. Find the ideal $I(V)$ of the algebraic subset of $\mathbb{A}^2$ defined by the equations $X^3 = Y^3 = XY(X + Y) = 0$. Does $X + Y$ belong to $I(V)$?

1.17. Find the radical of the ideal in $k[X, Y]$ generated by the polynomials X^2Y and XY^3.

1.18. Let $X_1, X_2 \subset \mathbb{A}^n$ be algebraic sets. Show that
(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) \supset \sqrt{I(X_1) + I(X_2)}$, with equality if k is algebraically closed.
Show by example that taking the radical in (ii) is in general necessary, i.e. find algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$, even if k is algebraically closed.. Can you see geometrically what it means if we have inequality here?

1.19. Let V and W be algebraic sets in $\mathbb{A}^n(k)$, with $V \subset W$. Show that each irreducible component of V is contained in some irreducible component of W.

1.20. If $V = V_1 \cup \cdots \cup V_r$ is the decomposition of an algebraic set into irreducible components, show that $V_i \not\subset \cup_{j \neq i} V_j$.

1.21. Show that $f = Y^2 + X^2(X - 1)^2 \in \mathbb{R}[X, Y]$ is an irreducible polynomial, but $V(F)$ is reducible.

1.22. i) Prove Gauß' Lemma: if A is a UFD and F, $G \in A[X]$ are polynomials with coefficients in A, then a prime element of A that is a common factor of the coefficients of the product FG is a common factor of the coefficients of F or G.
ii) Let A be a UFD. Prove that $A[X]$ is a UFD. For this you need to compare factorisations in $A[X]$ with factorisations in $Q(A)[X]$, where $Q(A)$ is the field of fractions of A, using Gauß' lemma to clear denominators.
iii) A field K is a UFD. Prove by induction on n that $K[X_1, \dots, X_n]$ is a UFD.

1.23. Let $X \subset \mathbb{A}^3$ be the algebraic set given by the equations $X_1^2 - X_2X_3 = X_1X_3 - X_1 = 0$. Find the irreducible components of X. What are their prime ideals? (Don't let the simplicity of this exercise fool you. It is in general very difficult to compute the irreducible components of the zero locus of given equations, or even to determine if it is irreducible or not.)

1.24. Show that the primary ideal $(X^2, XY, Y^2) \subset k[X,Y]$ is not irreducible. Write it as finite intersection of irreducible ideals.

1.25. Give an example of a continuous map in the Zariski topology which is not a regular map.

1.26. Let K be an algebraically closed field of characteristic $p > 0$. Consider the Frobenius map $F\colon K^n \to K^n$ given by $F(x_1, \dots, x_n) = (x_1^p, \dots, x_n^p)$. Show that F is regular and bijective, but not an isomorphism.

1.27. Let $f \in k[V]$, V a variety in $\mathbb{A}^n(k)$. Define the graph of f as

$$G(f) = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}^{n+1} \mid (a_1, \dots, a_n) \in V \text{ and } a_{n+1} = f(a_1, \dots, a_n)\} .$$

Show that $G(f)$ is an affine variety, and that the map $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, f(a_1, \dots, a_n))$ defines an isomorphism of V with $G(f)$. (Projection gives the inverse.)

1.28. Let k be an algebraically closed field. Let I be an ideal in $k[X_1, \dots, X_n, Y_1, \dots, Y_m]$. The inclusion $k[X_1, \dots, X_n] \subset k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ defines the projection map $p\colon \mathbb{A}^{n+m} \to \mathbb{A}^n$. Let $J = I \cap k[X_1, \dots, X_n]$. Show that $V(J)$ is the Zariski closure of the image $p(V(I))$ of $V(I)$ under the projection p.

1.29. Prove that the variety defined by the equation $XY - 1 = 0$ is not isomorphic to the affine line $A^1(k)$.

1.30. Let $f\colon \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map. Are the following statements true or false:
(i) if $V \subset \mathbb{A}^n$ is an algebraic set, then the image $f(V) \subset \mathbb{A}^m$ is an algebraic set.
(ii) if $W \subset \mathbb{A}^m$ is an algebraic set, then the inverse image $f^{-1}(W) \subset \mathbb{A}^n$ is an algebraic set.

1.31. Let $f\colon V \to W$ be a polynomial map between affine varieties, and let $f^*\colon k[W] \to k[V]$ be the corresponding map of k-algebras. Which of the following statements are true?
(i) If $P \in V$ and $Q \in W$, then $f(P) = Q$ if and only if $(f^*)^{-1}(I(P)) = I(Q)$.
(ii) f^* is injective if and only if f is surjective.
(iii) f^* is surjective if and only if f is injective.
(iv) f is an isomorphism if and only if f^* is an isomorphism.
If a statement is false, is there maybe a weaker form of it which is true?

1.32. Prove that a polynomial map $f\colon V \to W$ between affine varieties V and W is an isomorphism of V with a subvariety $f(V)$ of W if and only if the induced map $f^*\colon k[W] \to k[V]$ between coordinate rings is surjective.

1.33. Let $V = V(XY - ZT) \subset \mathbb{A}^4(k)$ and let x, y, z and t be the classes of the coordinate functions in $k[V]$. Consider the function $f \in k(V)$, $f = \frac{x}{z} = \frac{t}{y}$. Show that it is impossible to write $f = \frac{a}{b}$, where $a, b \in k[V]$, and $b(P) \neq 0$ for every P where f is defined. Show that the pole set of f is exactly $\{(x,y,z,t) \mid y = 0 \text{ and } z = 0\}$.

1.34. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be the function $f(X) = (X, 0)$, and let $g\colon \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ be the rational function $g(X,Y) = \frac{X}{Y}$. Show that the composite $g \circ f$ is not defined anywhere.
What is the largest subset of the function field $k(\mathbb{A}^2) = k(X,Y)$ on which f^* is defined?

1.35. Describe the regular map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$, $f(X,Y) = (X, XY)$ geometrically. What is its (rational) inverse? Determine $f^{-1}(C)$, where C is the curve $V(X^p - Y^q)$, $p, q > 1$ relatively prime.

1.36. Let $\mathcal{O}_{V,P}$ be the local ring of a variety V at a point P. Show that there is a natural one-to-one correspondence between the prime ideals in $\mathcal{O}_{V,P}$ and the subvarieties of V that pass through P.

1.37. Consider the following algebraic sets in $\mathbb{A}^2(\mathbb{C})$:

$$\begin{aligned} V &= V(XY) \\ W &= V(Y^2 - X^2 - X^3) \end{aligned}$$

The first curve consists of the coordinate axes, and the second is a cubic curve with a double point at the origin.
a) Show that the local ring of V at a point $P = (0, y)$, $y \neq 0$ is isomorphic to $\mathbb{C}[t]_{(t)}$, the local ring of the affine line at a point. So the second component V is not 'visible' in the local ring $\mathcal{O}_{V,P}$.
b) Show that the local ring of W at the origin is an integral domain, but that the local ring of Y at the origin is not. (This means that although locally in the Euclidean topology both curves look the same at the origin, this is not true in the Zariski topology.)

1.38. An affine variety V is called *rational* if $k(V) \cong k(X_1, \dots, X_n)$ for some n. Show that an irreducible conic $V(f) \subset \mathbb{A}^2(k)$ is rational if and only if $V(f)$ contains a point, defined over k.

1.39. Let $X, Y \subset \mathbb{A}^4$ be varieties defined by $X = \{(t, t^2, t^3, 0) \mid t \in k\}$, $Y = \{(0, u, 0, 1) \mid u \in \mathbb{C}\}$. The join variety of X and Y is the set

$$J(X,Y) = \bigcup_{P \in X, Q \in Y} \overline{PQ} \subset A^4\ ,$$

where $\overline{PQ}$ is the line through P and Q. Is $J(X,Y)$ an affine (or quasi-affine) variety?

1.40. Let K be an algebraically closed field. Show that $W = A^2(K) \setminus (0)$ is not an (abstract) affine variety.
Hint: show that $\mathcal{O}(W) = K[X,Y]$.

1.41. Consider the action of $\mathbb{Z}_2 = \{\pm 1\}$ on $\mathbb{A}^2(\mathbb{C})$, given by $\varepsilon\colon (X,Y) \mapsto (-X, -Y)$. Let $V = \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_2$ be the quotient. Define $\mathbb{C}[V]$ as $\mathbb{C}[X,Y]^{\mathbb{Z}_2}$, the algebra of $\mathbb{Z}_2$-invariant polynomials: $p(-X,-Y) = p(X,Y)$. Show that V is an abstract affine variety.