MMA320 Introduction to Algebraic Geometry
Exercises for Chapter 3

3.1. Let $f(X)$ and $G(X)$ be polynomials in $A[X]$, A an UFD. Let $f(X) = a_0(X - \alpha_1)\dots(X-\alpha_m)$ and $g(X) = b_0(X-\beta_1)\dots(X-\beta_n)$ with $b_0 \neq 0$ be the factorisation of the polynomials into linear factors (in an extension of the field of fractions $Q(A)$). Show that $R(f,g) = a_0^n b_0^m \prod_{i=1}^m \prod_{j=1}^n (\alpha_i - \beta_j) = a_0^n \prod_{i=1}^m g(\alpha_i) = (-1)^{mn} b_0^m \prod_{j=1}^n f(\beta_j)$.

3.2. Show that $R(f,gh) = R(f,g)R(f,h)$. Use this to show that $I_P(C,E) = I_P(C,D) + I_P(C,E)$.

3.3. Show that $R(f, g+af) = R(f,g)$ (suppose $\deg f \le \deg g$). Let $C=(f)$, $D=(g)$ and $E=(g+af)$ be plane curves. Show that $I_P(C,E) = I_P(C,D)$ for all $P \in C \cap D$.

To compute intersection multiplicities, one can use the results of the previous two exercises together with proposition 3.15.

3.4. Let C_1, $C_2 \subset \mathbb{P}^2(K)$, K algebraically closed of characteristic zero, be cubic curves with homogeneous equations

$$\begin{aligned} C_1\colon &\quad (X+Z)^3 + 3Y^3 - Z^3 = 0,\\ C_2\colon &\quad X^3 + X^2(Y+Z) + XZ^2 + Y^3 = 0. \end{aligned}$$

Compute the intersection points of C_1 and C_2 with their multiplicities.

3.5. Find all points of intersection of the following pairs of curves, and the intersection numbers at these points:
(a) $Y^2Z - X(X-2Z)(X+Z)$ and $Y^2 + X^2 - 2XZ$.
(b) $(X^2+Y^2)Z + X^3 + Y^3$ and $X^3 + Y^3 - 2XYZ$.
(c) $Y^5 - X(Y^2 - XZ)^2$ and $Y^4 + Y^3Z - X^2Z^2$.
(d) $(X^2+Y^2)^2 + 3X^2YZ - Y^3Z$ and $(X^2+Y^2)^3 - 4X^2Y^2Z^2$.

3.6. Suppose the intersections of the opposite sides of a hexagon lie on a straight line. Show that the vertices lie on a conic.

3.7. Let $C=(f)$ be a projective plane curve. Show that a point P is a multiple point of C if and only if $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0$.

3.8. Let P be a simple point on $C=(f)$. Show that the tangent line to C at P has the equation $\frac{\partial f}{\partial X}(P)X + \frac{\partial f}{\partial Y}(P)Y + \frac{\partial f}{\partial Z}(P)Z = 0$.

3.9. Show that $m_P(\frac{\partial f}{\partial X}) \ge m_P(f) - 1$ for all f and P with $f(P)=0$.

3.10. Show that the following curves are irreducible; find their multiple points, and the multiplicities and tangents at the multiple points.
(a) $XY^4 + YZ^4 + XZ^4$.
(b) $X^2Y^3 + X^2Z^3 + Y^2Z^3$.
(c) $Y^2Z - X(X-Z)(X-\lambda Z)$, $\lambda \in k$.
(d) $X^n + Y^n + Z^n$, $n > 0$.

3.11. Let $C = (f)$ be an irreducible curve.
(a) Show that one of $\frac{\partial f}{\partial X}$, $\frac{\partial f}{\partial Y}$ and $\frac{\partial f}{\partial Z}$ is not the zero polynomial.
(b) Show that C has only finitely many multiple points.

3.12. Prove that an irreducible cubic is either nonsingular or has at most one double point.

3.13. Show that every nonsingular curve in a projective plane over an algebraically closed field is irreducible. Is this true for affine curves?

3.14. A line L is a component of the tangent cone of a curve C at a point P if and only if $I_P(C, L) > m_P(C)$.

3.15. ($\mathrm{char}(k) = 0$) Let C be an irreducible curve of degree n in $\mathbb{P}^2$. Suppose $P \in \mathbb{P}^2$, with $m_P(C) = r \geq 0$. Then for all but a finite number of lines L through P, L intersects C in $n - r$ distinct points other than P.

3.16. ($\mathrm{char}(k) = p > 0$) Let $C = V(X^{p+1} - Y^p Z)$, $P = (0 : 1 : 0)$. Find $L \cap C$ for all lines L passing through P. Show that every line that is tangent to C at a simple point passes through P.

3.17. If a curve C of degree n has a point P of multiplicity n, show that C consists of n lines through P (not necessarily distinct).

3.18. Let $f(x, y)$ be the affine equation of a real or complex plane curve, and $p = (a, b)$ a point on it; suppose that $\frac{\partial f}{\partial y}(p) \neq 0$, so by the implicit function theorem $y = \varphi(x)$ in a neighbourhood of p. Prove that p is an inflection point (in the sense that $\varphi''(a) = 0$) if and only if:

$$\begin{vmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{vmatrix} = 0.$$

(Hint: differentiate $f(x, \varphi(x)) \equiv 0$ twice. Compute also the determinant.)
Use Euler's formula and $f(a, b) = 0$ to translate this condition into the condition on the vanishing of the Hessian of the associated homogeneous polynomial $F(X, Y, Z)$.

3.19. Let $C \subset \mathbb{P}^2(k)$, $\mathrm{char}\, k \neq 2$, be a plane cubic with inflection point P. Prove that a change of coordinates can be used to bring C in the normal form $Y^2 Z = X^3 + aX^2 Z + bXZ^2 + cZ^3$.
Hint: take coordinates such that $P = (0 : 1 : 0)$, and its tangent is $Z = 0$; get rid of the linear term in Y by completing the square.

3.20. A real plane cubic with one inflection point has two other inflection points.
Hint: use the result of the previous exercise in affine coordinates, express y as function of x and show that $y''(x)$ has to have a zero.

3.21. Find the inflection points of the singular cubics $ZY^2 = X^2(X + Z)$ and $Y^2 Z = X^3$, both in $\mathbb{P}^2(\mathbb{R})$ and in $\mathbb{P}^2(\mathbb{C})$.

3.22. Let P and Q be inflection points on a cubic curve C. Show that the third intersection point of the line $\overline{PQ}$ with C is also an inflection point.
Hint: use coordinates in which $P = (0 : 1 : 0)$, $Q = (0 : 0 : 1)$ and the flexes are $Y = 0$ and $Z = 0$.

3.23. Consider the cubic curve C in $\mathbb{P}^2(\mathbb{C})$ with equation:

$$X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0,$$

where $\lambda^3 \neq 1$. Find its inflection points. Compute the inflectional tangents. What are all lines joining inflection points?

3.24. Give an example (in char p) of an irreducible curve in the projective plane with identically vanishing Hessian, but where not all points are inflection points.

3.25. For which points P on a nonsingular cubic C does there exist a nonsingular conic that intersects C only at P?

3.26. Let C be the cubic curve in $\mathbb{P}^2(\mathbb{Q})$ with affine equation:

$$Y^2 = X^3 - 2X^2 + 1\,.$$

Take $(0:1:0)$, the inflection point at infinity, as neutral element of the group law. Compute all multiples $2p$, $3p$, $4p$, $5p$, $\ldots$, of the point $p = (0,1)$.

3.27. The irreducible cubic curve $C = (Y^2Z - X^3)$ has a cusp in $Q = (0:0:1)$ and an inflection point in $(0:1:0)$. Show that the set $C^{\mathrm{reg}} = C \setminus Q$ has a group structure isomorphic to the additive group K^+ of the field K.
Hint: take the inflection point as neutral element and use a suitable parametrisation of C^{reg}.