MMA320 Introduction to Algebraic Geometry
Exercises for Chapter 5

5.1. Show that if k is a field of characteristic $p > 0$, and $f \in k[X_1, \ldots, X_n]$, then $\frac{\partial f}{\partial X_i} = 0$ if and only if $f \in k[X_1, \ldots, X_i^p, \ldots, X_n]$. Conclude that all the $\frac{\partial f}{\partial X_i}$ are zero if and only if $f \in k[X_1^p, \ldots, X_n^p]$.

5.2. Let $V \subset \mathbb{P}^n$ be a projective variety of dimension d. Show that there exists disjoint linear subspaces L_d and L'_{n-d-1} of dimension d and $n-d-1$, such that projection from L'_{n-d-1} maps V onto L_d and over every point of L_d lie only finitely many points of V.

5.3. Let $V \subset \mathbb{P}^n$ be a smooth variety. Show that projection from a point $P \in \mathbb{P}^n$ outside V induces an isomorphism from V onto its image if and only if for all lines l through P the intersection multiplicity of V and l is at most 1, that is no line intersects V in two points, or is a tangent.

5.4. Show that a smooth variety $V \subset \mathbb{P}^n$ of dimension d can be projected isomorphically onto a smooth variety in $\mathbb{P}^{2d+1}$.

5.5. Prove that the intersection of an irreducible hypersurface $V(F) \subset \mathbb{A}^n$ (not a hyperplane) with the tangent hyperplane $T(V)_p$ in a nonsingular point $p \in V$ is singular at p. (Here one defines the intersection by the ideal (F, L), where L is the linear equation for the tangent hyperplane.)
Show that this need not be true if one defines the intersection with reduced structure, i.e., by the radical of the ideal (F, L).

5.6. Find the singular points of the surface $Z^2 = X^2Y$ in $\mathbb{A}^3$. Make a sketch of the real points of this surface. Describe also its projective closure.

5.7. Find the singular points of the Steiner quartic

$$V(X_2^2X_3^2 + X_1^2X_3^2 + X_1^2X_2^2 - X_0X_1X_2X_3)$$

in $\mathbb{P}^3$.

5.8. Determine the singular locus of $V(I) \subset \mathbb{A}^7$, where I is the ideal

$$(A_1D_1 - SA_2D_2,\ B_1D_1 - SB_2D_2,\ A_1D_2 + A_2D_1,\ B_1D_2 + B_2D_1).$$

Find the irreducible components.

5.9. Find an ideal of $k[X_1, \ldots, X_n]$ which admits minimal sets of generators differing in their number of elements; here minimal means that no proper subset generates.