MMA320 Introduction to Algebraic Geometry

Exercises for Chapter 6

6.1. Suppose a hypersurface $V = (f)$ of degree $d > 1$ in $\mathbb{P}^n$ contains a linear subspace of dimension $r \geq n/2$. Show that V is singular.

6.2. Prove that there is at least one line through a singular point of a cubic surface (and 'in general' 6). Let $X \subset \mathbb{P}^4$ be a non-singular cubic threefold. Show that through every point there is at least one line.

6.3. Show that the surface $V(X^n + Y^n + Z^n + T^n) \subset \mathbb{P}^3$ contains exactly $3n^2$ lines ($\operatorname{char} k = 0$ and $k = \bar{k}$).

6.4. Find all 27 lines on Clebsch' diagonal surface:

$$\begin{aligned} X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 &= 0 \\ X_0 + X_1 + X_2 + X_3 + X_4 &= 0 \end{aligned}$$

All lines are real, 15 are easy to see as intersections with coordinate hyperplanes.

6.5. Find the singular points of the cubic surface

$$X_0X_1X_2 + X_0X_1X_3 + X_0X_2X_3 + X_1X_2X_3 = 0 \,.$$

Determine all lines on the surface.

6.6. Find the singular point of the cubic surface

$$ZX^2 - Y^3 + TZ^2 = 0 \,.$$

Determine all lines on the surface.

6.7. Determine the number of lines on the singular cubic surface

$$XYZ - T^3 = 0 \,.$$

6.8. Let $P_1, \ldots, P_6$ be 6 points on a nondegenerate conic $C = (f) \subset \mathbb{P}^2$. The linear system $\mathbb{P}(S_3(P_1, \ldots, P_6))$ of cubics through these six points gives rise to a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$. Compute the equation of the image.

6.9. Let π be the skew projection of a smooth cubic surface from two skew lines l and m on a plane through the transversal l_1. Show that π is a regular map and determine which lines are blown down.

6.10. Let S be the cubic surface, which is the image of the rational map determined by the linear system of cubics in the plane through 6 points, no three on a line, not all on a conic. Describe the 27 lines in terms of points and curves in the plane.