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MMA340 - Analytic Number Theory
Spring semester 2019

Assignment 1

The following is the first of the three homework assignments. The deadline for handing in your solutions is **Friday February 8 (2019) at 10.00**. You should aim to provide complete solutions with proper justifications. However, all partial solutions will be considered and will be given appropriate partial credits.

1. *The Ramanujan sum* $c_q(n)$ is defined by the formula

$$c_q(n) := \sum_{\substack{1\le m\le q\\(m,q)=1}} e^{2\pi i m n/q} \qquad (q, n \in \mathbb{N}).$$

a) Prove that

$$c_q(n) = \sum_{d|(n,q)} d\mu\left(\frac{q}{d}\right).$$

Hint: Consider first the completed sum $\sum_{1\le m\le q} e^{2\pi i m n/q}$.

b) Show that

$$\mu(n) = \sum_{\substack{1\le m\le n\\(m,n)=1}} e^{2\pi i m/n}.$$

c) Prove that

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\phi(q)}{\phi\left(q/(q,n)\right)}.$$

2. Choose one of Problem 2a and Problem 2b and submit only your solution to this chosen problem.

a) An arithmetic function f is called *periodic* if there exists a number $k \in \mathbb{N}$ (called the period of f) such that $f(n+k) = f(n)$ for all $n \in \mathbb{N}$. Prove that if f is completely multiplicative and periodic with period k, then the values of f are either 0 or roots of unity.

b) Let $f : \mathbb{N} \to \mathbb{N}$ be a multiplicative and strictly increasing arithmetic function. Assume that $f(2) = 2$. Prove that $f(n) = n$ for all $n \in \mathbb{N}$.

3. Let f be an arithmetic function and suppose that $\sum_{n\le N} f(n) \sim N^2$ as $N \to \infty$. Prove that

$$\sum_{n\le N} f(n)(N-n)^2 \sim \tfrac{1}{6}N^4 \qquad \text{as } N \to \infty.$$

4. Let $\alpha \geq 0$ and $\delta < 1$ be fixed. Let f be an arithmetic function and suppose that $f(n) = O(n^\alpha)$ and

$$A_f(x) = \sum_{n \leq x} f(n) = O(x^\delta).$$

Define a second arithmetic function g by

$$g(n) = \sum_{d|n} f(d).$$

Prove that

$$\sum_{n \leq x} g(n) = cx + O\left(x^{\frac{1+\alpha}{2+\alpha-\delta}}\right)$$

for some constant c. Determine the constant c explicitly.

GOOD LUCK!