University of Gothenburg
Department of Mathematical Sciences
Anders Södergren

MMA340 - Analytic Number Theory
Spring semester 2019

Assignment 2

The following is the second of the three homework assignments. The deadline for handing in your solutions is **Friday February 22 (2019) at 10.00**. You should aim to provide complete solutions with proper justifications. However, all partial solutions will be considered and will be given appropriate partial credits.

1. Let $P \in \mathbb{Z}[x]$ be a nonzero polynomial of degree d.

   a) Prove that

$$\int_0^1 P(t)^{2n}\,dt \geq \frac{1}{\operatorname{lcm}[1,2,\ldots,2dn+1]} = e^{-\psi(2dn+1)}.$$

   b) Suppose now that $d \geq 1$. Show that

$$\psi(2dn+1) > 2n \log\left(\min_{0\leq t\leq 1} \frac{1}{|P(t)|}\right)$$

   and experiment with this inequality in order to get as strong a lower bound on ψ as possible.

2. Choose one of Problem 2a and Problem 2b and submit only your solution to this chosen problem.

   a) Prove, using Dirichlet series, that

$$\Lambda(n)\log n + \sum_{d|n} \Lambda(d)\Lambda\Big(\frac{n}{d}\Big) = \sum_{d|n} \mu(d)\left(\log\Big(\frac{n}{d}\Big)\right)^2$$

   holds for all $n \in \mathbb{N}$.

   b) Let f be the indicator function of those $n \in \mathbb{N}$ whose base 10 representation does not contain the digit 9, and let $F(s)$ denote the Dirichlet series associated to f. Determine the abscissa of convergence and the abscissa of absolute convergence of $F(s)$.

3. Let

$$F(s) := \sum_{n=1}^{\infty} \frac{1}{\phi(n)^s},$$

   where ϕ is Euler's ϕ-function.

a) Prove that the series $F(s)$ converges absolutely and uniformly on compact subsets of $\{s : \sigma > 1\}$.

b) Show that

$$\frac{F(s)}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{(p-1)^s}\right), \tag{0.1}$$

and prove that this product converges absolutely and uniformly on compact subsets of $\{s : \sigma > 0\}$.

c) Define $f(n) = \#\{m : \phi(m) = n\}$. Then $F(s)$ is the Dirichlet series associated to f. Does the expression in (0.1) provide an Euler product representation for this Dirichlet series?

4. This is an exercise in elementary techniques. No tools form complex analysis (or the general theory of Dirichlet series) are needed/allowed.

a) Prove that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{A}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

for some suitable positive constant A, as $x \to \infty$.

b) Upgrade the result in part a) by proving that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

as $x \to \infty$.

Hint: Consider the series $\sum_p p^{-s}$ as $s \to 1+$.

GOOD LUCK!