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MMA340 - Analytic Number Theory
Spring semester 2019

Assignment 3

The following is the third of the three homework assignments. The deadline for handing in your solutions is **Friday March 8 (2019) at 10.00**. You should aim to provide complete solutions with proper justifications. However, all partial solutions will be considered and will be given appropriate partial credits.

1. **a)** For $d \in \mathbb{Z} \setminus \{0\}$ we let $\chi_d(n)$ denote the Dirichlet character induced by the Kronecker symbol $\left(\frac{d}{n}\right)$. According to Davenport (page 40) the only nontrivial real primitive characters which have conductor a power of 2 are $\chi_{-4}(n)$, $\chi_8(n)$ and $\chi_{-8}(n)$. Create a table with all values of these three characters for $1 \leq n \leq 8$. Prove that the values in the table are correct.

   b) Let χ be a nontrivial Dirichlet character modulo q. Show that $\left|\sum_{n \leq x} \chi(n)\right| \leq q$ for all $x \geq 1$.

   c) Let χ be a nontrivial Dirichlet character modulo q. Prove that $L(s, \chi)$ converges in the half-plane $\sigma > 0$.

   d) Let χ^* be a Dirichlet character modulo q^*. If q is a multiple of q^*, then

$$\chi(n) := \begin{cases} \chi^*(n) & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1, \end{cases}$$

   is a Dirichlet character modulo q. (We say that χ is induced by χ^*.) Prove that

$$L(s, \chi) = L(s, \chi^*) \prod_{p \mid q} \left(1 - \chi^*(p) p^{-s}\right)$$

   in the half-plane $\sigma > 1$.

2. Choose one of Problem 2a and Problem 2b and submit only your solution to this chosen problem.

   a) Determine $\zeta(0)$ and $\zeta(-1)$.

   b) Determine $\zeta'(0)$.

3. **a)** Prove that for all sufficiently large x the interval $[0, x]$ contains more prime numbers than the interval $(x, 2x]$.

b) Upgrade the result in part a) as follows: Let $A > 0$ be fixed. Prove that if $n < (\log x)^A$ and x is sufficiently large then each of the intervals

$$[0, x], (x, 2x], \ldots, ((n-1)x, nx]$$

contains more primes than the succeeding interval.

4. Prove that the Riemann hypothesis is equivalent to the estimate $\psi(x) = x + O\left(x^{1/2}(\log x)^2\right)$ as $x \to \infty$.

GOOD LUCK!