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MMA340 - Analytic Number Theory
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Exercise sheet 7

1. Prove that for every nonzero $d \equiv 0$ or $1 \mod 4$, the function $\chi_d(n) := \left(\frac{d}{n}\right)$ defined in terms of the Kronecker symbol is a Dirichlet character modulo $|d|$.

2. Let $q \in \mathbb{N}$ and let $g_1, g_2, \ldots, g_r$ be the generators of $(\mathbb{Z}/q\mathbb{Z})^\times$ of orders $h_1, h_2, \ldots, h_r$ mod q. Let $\nu = (\nu_1, \ldots, \nu_r)$ be an r-tuple of integers satisfying $0 \leq \nu_j < h_j$ for $1 \leq j \leq r$. Define the function $\chi : \mathbb{Z} \to \mathbb{C}$ by
$$\chi(n) = \begin{cases} 0 & \text{if } (n,q) > 1, \\ \prod_{i=1}^r e^{2\pi i \nu_i \mu_i / h_i} & \text{if } n \equiv \prod g_i^{\mu_i} \mod q. \end{cases}$$
Prove that χ is a Dirichlet character.

3. Prove that for any $n \in \mathbb{Z}$ satisfying $(n,q) = 1$ and $n \not\equiv 1 \mod q$ there exists a Dirichlet character χ modulo q with $\chi(n) \neq 1$.

4. Find all Dirichlet characters modulo 12 and 15.

5. Let $q \in \mathbb{N}$. Prove that for any character χ modulo q we have
$$\sum_{n=1}^{q} \chi(n) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases}$$

6. Let $q \in \mathbb{N}$. Prove that for any characters χ_1 and χ_2 modulo q we have
$$\sum_{n=1}^{q} \chi_1(n)\overline{\chi_2(n)} = \begin{cases} \phi(q) & \text{if } \chi_1 = \chi_2, \\ 0 & \text{otherwise.} \end{cases}$$

7. Let $q \in \mathbb{N}$. Prove that for any $a, n \in \mathbb{Z}$ we have
$$\sum_{\chi \mod q} \chi(n)\overline{\chi(a)} = \begin{cases} \phi(q) & \text{if } n \equiv a \mod q \text{ and } (a,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

8. Let $q \in \mathbb{N}$. Prove that all characters modulo q are real if and only if $q \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.

   Comment: This might be a more advanced exercise (depending on your background).

9. Let χ be a nontrivial Dirichlet character modulo q. Show that
$$\sum_{n > x} \frac{\chi(n)}{n} = O\left(\frac{1}{x}\right).$$

10. Let χ be a real-valued Dirichlet character modulo q and define the arithmetic function f by

$$f(n) := \sum_{d|n} \chi(d).$$

a) Show that $f(1) = 1$ and that $f(n) \geq 0$ for all $n \in \mathbb{N}$.

b) Prove that $f(n) \geq 1$ when n is a perfect square.

11. Let $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ be such that $(a, q) = 1$. Prove that

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \log L(s, \chi) = \sum_{p \equiv a \pmod q} \frac{1}{p^s} + O(1)$$

in $\sigma > 1$.

12. Prove that

$$\frac{L'(s, \chi)}{L(s, \chi)} = -\sum_{n=1}^{\infty} \frac{\chi(n)\Lambda(n)}{n^s}$$

in $\sigma > 1$.

13. Show that

$$\frac{L'(s, \chi)}{L(s, \chi)} = -s \int_1^{\infty} \psi(x, \chi) x^{-s-1} \, dx$$

in $\sigma > 1$, where

$$\psi(x, \chi) := \sum_{n \leq x} \chi(n)\Lambda(n).$$

14. **a)** Show that for any $n > 0$, we have

$$\Gamma(s) = n^s \int_0^1 x^{n-1} \left(\log(x^{-1})\right)^{s-1} \, dx$$

in $\sigma > 0$.

b) Let χ be a Dirichlet character modulo $q \geq 2$. Show that

$$\sum_{n=1}^{\infty} \chi(n) x^n = (1 - x^q)^{-1} \sum_{n=1}^{q-1} \chi(n) x^n$$

for all $|x| < 1$.

c) Let χ be a nontrivial Dirichlet character modulo q. Prove that

$$\Gamma(s) L(s, \chi) = \int_0^1 (1 - x^q)^{-1} \sum_{n=1}^{q-1} \chi(n) x^{n-1} \left(\log(x^{-1})\right)^{s-1} \, dx$$

in $\sigma > 0$.

d) Let χ be a nontrivial Dirichlet character modulo q. Show that $L(s, \chi)$ can be continued analytically to an *entire* function.

15. For any Dirichlet character χ modulo q, the corresponding *Gauss sum* $\tau(\chi)$ is defined by

$$\tau(\chi) := \sum_{m=0}^{q-1} \chi(m) e^{2\pi i m/q}.$$

a) Prove that if $(n, q) = 1$, then

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{h=0}^{q-1} \overline{\chi}(h) e^{2\pi i h n/q}.$$

Comment: Here you may suppose that $\tau(\overline{\chi}) \neq 0$. This is a consequence of Exercise 16.

b) Let χ be a primitive Dirichlet character modulo q. Prove that for any $a, b \in \mathbb{Z}$, we have

$$\frac{1}{q} \sum_{c=0}^{q-1} \chi(ac+b) = \begin{cases} \chi(b) & \text{if } q | a, \\ 0 & \text{otherwise.} \end{cases}$$

Hint: For the second (and more difficult) case, recall the essential trick in the proof of Theorem 1 from Lecture 19.

c) Let χ be a primitive Dirichlet character modulo q. Use the formula in part b) to prove that if $(n, q) > 1$, then

$$\sum_{h=0}^{q-1} \overline{\chi}(h) e^{2\pi i h n/q} = 0.$$

Conclude that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{h=0}^{q-1} \overline{\chi}(h) e^{2\pi i h n/q}$$

holds for all n.

16. **a)** Let χ be a primitive Dirichlet character modulo q. Prove that $|\tau(\chi)| = \sqrt{q}$.

Hint: Here it might be useful to review the formulas we have studied for Ramanujan sums.

b) Prove that if we in addition suppose that $\chi(-1) = 1$, then $\tau(\chi)\tau(\overline{\chi}) = q$.

c) Prove that if we in addition suppose that $\chi(-1) = -1$, then $\tau(\chi)\tau(\overline{\chi}) = -q$.

17. Let χ be a nonprincipal Dirichlet character and define $f(n) = \sum_{d|n} \chi(d)$ for all $n \in \mathbb{N}$. Use Dirichlet's hyperbola method to show that

$$\sum_{n \leq x} \frac{f(n)}{n^{1/2}} = 2L(1, \chi) x^{1/2} + O(1).$$

18. Let χ be a real nonprincipal Dirichlet character. Use the previous exercise to prove that $L(1, \chi) \neq 0$.

    Comment: Note that this gives an alternative proof of Theorem 5 (Case 2). You can also try to show that $L(1, \chi) > 0$.