

Ordinary differential equations (MMA420/MAN460/TMA013)

Time: 2008-08-19, 08.30–13.30

Tools: No calculator or handbook is allowed

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Grades: Each problem gives 3 points, except problem 7, which gives 6 points. For MMA420/MAN460 grades are G (12-17 points) and VG (18-24 points). For TMA013 grades are 3 (12-14 points), 4 (15-17 points) and 5 (18-24 points).

Please write your name and personal identification number on each sheet of paper handed in.

- 1 Determine all equilibrium points (jämviktpunkter) to the system

$$\begin{cases} x' = x^2 - 1 - y, \\ y' = (x^2 + 1)y, \end{cases}$$

and investigate whether they are stable.

- 2 Solve the initial value problem

$$\begin{cases} x' = x + y, & x(0) = 1, \\ y' = -x + 3y, & y(0) = 0. \end{cases}$$

- 3 Rewrite the differential equation

$$x'' + (x')^3 + x^3 = 0$$

as a first order system. Show that, for suitable constants A and B , $E(x, y) = Ax^4 + By^2$ is a Lyapunov function.

- 4 Let $A = (a_{jk})_{1 \leq j, k \leq n}$ be an $n \times n$ matrix. Denote by $\|A\|$ the usual matrix norm and by $\|A\|_{\max}$ the maximum norm; that is,

$$\|A\| = \max_{|\mathbf{x}|=1} |A\mathbf{x}|, \quad \|A\|_{\max} = \max_{1 \leq j, k \leq n} |a_{jk}|.$$

Show that

$$\|A\|_{\max} \leq \|A\| \leq n\|A\|_{\max}.$$

- 5 Determine Green's function for the boundary value problem

$$y'' - 4y' + 4y = f, \quad y(0) = y'(1) = 0,$$

and use it to express the solution y as an integral.

- 6 Consider the initial value problem

$$\begin{cases} x' = -x^2 - y, & x(0) = 0, \\ y' = y^2, & y(0) = -1. \end{cases}$$

What happens to the solution as $t \rightarrow \infty$?

- 7 (a) Formulate an existence and uniqueness theorem for initial value problems. If you use terminology that has been introduced during the course, please explain it.

(b) Prove the theorem stated in part (a).

N. B. A complete solution to Problem 7 gives 6 points.

Good luck!

Hjalmar