Exercises for Friday, March 6

1. Determine Green's function for the problem

$$y'' + y = f, \qquad y(0) = y'(\pi) = 0.$$

2. Find (as an integral) the solution to the boundary value problem

$$y'' + 3y' = f, \qquad y(0) = 1, \quad y'(1) = 6.$$

3. Determine Green's function for the problem

$$y^{(3)} = f, \qquad y(0) = y''(0) = y'(1) = 0.$$

4. Under what condition does the general first order boundary value problem

$$y'(x) + f(x)y(x) = g(x), \qquad Ay(\alpha) + By(\beta) = 0$$

have a unique solution for all g? When this is the case, write down a formula for the solution.

Answers (May contain typos!)

1. $G(x,\xi) = \begin{cases} -\sin\xi\cos x, & x \geq \xi, \\ -cos\xi \sin x, & x \leq \xi. \end{cases}$

2. $$y(x) = 1 + 2e^3 - 2e^{3-3x} + \int_0^1 G(x,\xi) f(\xi)\, d\xi,$$

   where
   $$G(x,\xi) = \begin{cases} (1-e^{3\xi})/3, & x \geq \xi, \\ (e^{3(x-\xi)} - e^{3\xi})/3, & x \leq \xi. \end{cases}$$

3. $G(x,\xi) = \begin{cases} (x^2+\xi^2-2x)/2, & x \geq \xi, \\ x(\xi-1), & x \leq \xi. \end{cases}$

4. The problem has a unique solution if $Ae^{-F(\alpha)} + Be^{-F(\beta)} \neq 0$, where $F' = f$. The solution is then given by

   $$y(x) = \frac{e^{-F(x)}}{Ae^{-F(\alpha)} + Be^{-F(\beta)}} \left(Ae^{-F(\alpha)} \int_\alpha^x e^{F(\xi)} g(\xi)\, d\xi - Be^{-F(\beta)} \int_x^\beta e^{F(\xi)} g(\xi)\, d\xi \right).$$