Exercises for Tuesday, January 27

1. Rewrite

$$x^{(3)} + t^2 x = 0, \qquad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1$$

as an initial value problem for a first-order linear system.

2. Which of the following functions are Lipschitz continuous on the interval $[-1, 1]$?
(a) $f(x) = |x|$; (b) $f(x) = \sqrt{|x|}$; (c) $f(x) = \dfrac{1}{x-2}$;
(d) $f(x) = x^2 \sin(1/x)$, $f(0) = 0$; (e), $f(x) = x \sin(1/x)$, $f(0) = 0$.

3. Show that the solution to the problem

$$x' = \sin(t) + \ln(1 + x^2), \qquad x(0) = 0$$

is uniquely defined for all $t \in \mathbb{R}$.

4. Find an interval where you can guarantee that the problem

$$x' = 1 + x^2 t^2, \qquad x(0) = 0,$$

has a unique solution.

5. Find an interval where you can gurantee that

$$x'' = \sqrt{x^2 + (x')^2}, \qquad x(0) = x'(0) = 0$$

has a unique solution.

6. Make a rough sketch of the vector field that is tangent to the solutions of the equation $x' = x - t$. Then solve the equation and sketch some solution curves.

7. Without solving the equation, find all orbits of the system

$$\begin{cases} x' = y(x^2 + 1), \\ y' = 2xy^2. \end{cases}$$

Sketch the phase portrait.

Answers

1. $\begin{cases} x' = y, & x(0) = 0 \\ y' = z, & y(0) = 0 \\ z' = -t^2 x, & z(0) = 1. \end{cases}$

2. All except (b) and (e).

3.

4. $|t| \leq 1/\sqrt{2}$.

5. $|t| \leq 1/\sqrt{2}$.

6.

7. The orbits are $y = C(x^2 + 1)$, except when $C = 0$. Moreover, all points on the x-axis are orbits.

A note for those who use Teschl's book:

Teschl's version of the global existence and uniqueness theorem (Theorem 2.5) is a bit more general than the one I did on the lecture. It is enough to know the following (Andersson–Böiers, Sats 1′), which follows by estimating the integrand in (2.23) by its maximum. Please contact me if you have questions about this.

Theorem: Suppose $f = f(t, \mathbf{x})$ is continuous and satisfies a Lipschitz condition (in the $\mathbf{x}$-variable) in the region $D = \{(t, \mathbf{x}); |t - t_0| \leq a, |\mathbf{x} - \mathbf{x}_0| \leq b\}$. Let $M = \max_D |f|$, and let $\alpha = \min(a, b/M)$. Then the initial value problem

$$\mathbf{x}' = f(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution in the interval $|t - t_0| \leq \alpha$.