Linearization

We follow an approach based on Lyapunov's method. We only give the backbones of the theory, and refer to the lecture and books for more explanation and examples.

We continue to study stability of equilibrium points for autonomous systems

$$x' = f(x).$$

For simplicity we assume that 0 is an equilibrium point, that is, $f(0) = 0$. If f is continuosly differentiable, we can write

$$f(x) = Ax + R(x),$$

where A is the Jacobian matrix with entries $(\partial f_i/\partial x_j)(0)$, and $R(x)/|x| \to 0$ as $x \to 0$. The following theorem shows that (when $\max(\mathrm{Re}(\lambda)) \neq 0$), the non-linear system $x' = f(x)$ has the same stability properties as the linearization $x' = Ax$.

Theorem:

- (a) If $\mathrm{Re}(\lambda) < 0$ for all eigenvalues of A, then 0 is asymptotically stable.
- (b) If $\mathrm{Re}(\lambda) > 0$ for some eigenvalue of A, then 0 is unstable.

In the remaining case $\max(\mathrm{Re}(\lambda)) = 0$, any one of the three cases asymptotically stable, non-asymptotically stable or unstable is possible. (As we have seen previously, in the linear case only the last two can happen.)

Lemma: Suppose that A is an $n \times n$ matrix such that $\mathrm{Re}(\lambda) < \beta$ for all eigenvalues. Then there exists a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ such that

$$\langle Ax, x \rangle < \beta \langle x, x \rangle, \qquad x \neq 0.$$

Proof: By writing $A = A_0 + \beta I$ one is immediately reduced to the case $\beta = 0$. When $\beta = 0$, we claim that

$$\langle x, y \rangle = \int_0^\infty e^{tA}x \cdot e^{tA}y \, dt \tag{1}$$

defines a scalar product with the desired properties. Here, $\cdot$ denotes the standard scalar product on $\mathbb{R}^n$.

Note first that, since each matrix entry of e^{tA} has the form $\sum_j p_j(t)e^{\lambda_j t}$, with p_j polynomials and λ_j eigenvalues, the integral (1) is convergent. We must show that $\langle\cdot,\cdot\rangle$ is a scalar product, that is, that it is bilinear, symmetric and positive definite. The first two properties are obvious. For positive definiteness, note that

$$\langle x, x\rangle = \int_0^\infty |e^{tA}x|^2\,dt \geq 0.$$

Since the integrand is continuous, $\langle x, x\rangle = 0$ only if $e^{tA}x = 0$ for all t, which gives $x = 0$.

Finally, we have

$$\begin{aligned}\langle Ax, x\rangle = \int_0^\infty e^{tA}Ax \cdot e^{tA}x\,dt &= \frac{1}{2}\int_0^\infty \frac{d}{dt}|e^{tA}x|^2\,dt \\ &= \frac{1}{2}\left[|e^{tA}x|^2\right]_{t=0}^\infty = -\frac{1}{2}|x|^2 < 0, \qquad x \neq 0.\end{aligned}$$

This completes the proof of the Lemma.

Proof of Theorem, part (a): Choose β such that $\mathrm{Re}(\lambda) < \beta < 0$ for all eigenvalues, and choose a scalar product $\langle\cdot,\cdot\rangle$ as in the Lemma. Let $\|\cdot\|$ denote the corresponding norm. We claim that $E(x) = \|x\|^2$ is a strict Lyapunov function. To see this, we write

$$\begin{aligned}\dot E(x(t)) = \frac{d}{dt}\langle x(t), x(t)\rangle = 2\langle x'(t), x(t)\rangle &= 2\langle Ax(t), x(t)\rangle + 2\langle R(x(t)), x(t)\rangle \\ \leq 2\beta\|x(t)\|^2 + 2\|R(x(t))\|\|x(t)\| &= 2\left(\beta + \frac{\|R(x(t))\|}{\|x(t)\|}\right)\|x(t)\|^2.\end{aligned}$$

Since all norms on $\mathbb{R}^n$ are equivalent, $|R(x)|/|x| \to 0$ implies $\|R(x)\|/\|x\| \to 0$ as $x \to 0$. Thus, $\dot E(x) < 0$ for $x \neq 0$ close enough to zero. This shows that $E(x)$ is a strict Lyapunov function, which implies asymptotic stability.

Proof of Theorem, part (b): After a change of variables, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where the eigenvalues of A_1 have positive real part and those of A_2 non-negative real part. That this is possible is clear from the Jordan canonical form; see AB Sats 2.9 for a direct proof. Let $\mathbb{R}^n = V_1 \oplus V_2$ be the corresponding decomposition of $\mathbb{R}^n$. Let $a > 0$ be a number such that $\mathrm{Re}(\lambda) > a > 0$

for all eigenvalues of A_1. Applying the Lemma to $-A_1$, we choose a scalar product $\langle\cdot,\cdot\rangle$ on V_1 such that

$$\langle x, A_1 x\rangle \geq a\langle x, x\rangle, \qquad x \in V_1.$$

Take b with $0 < b < a$. Applying the Lemma to A_2, we similarly choose a scalar product $[\cdot,\cdot]$ on V_2 such that

$$[x, A_2 x] \leq b[x, x], \qquad x \in V_2.$$

For $x \in \mathbb{R}^n$, write $x = x_1 + x_2$, where $x_i \in V_i$. We claim that

$$E(x) = \langle x_1, x_1\rangle - [x_2, x_2]$$

satisfies the conditions of the instability theorem in the previous lecture. To see this it is enough to show that $\dot{E}(x) > 0$ if $E(x) > 0$ and $x \neq 0$ is close enough to zero. Then, for ε small enough we can put $\Omega' = \{x;\ |x| < \varepsilon,\ E(x) > 0\}$ in the instability theorem.

Similarly as above, we write

$$\dot{E}(x) = 2(\langle A_1 x_1, x_1\rangle - [A_2 x_2, x_2]) + \rho(x) \geq 2(a\langle x_1, x_1\rangle - b[x_2, x_2]) + \rho(x),$$

where $\rho(x)/\|x\|^2 \to 0$ for any norm, as $x \to 0$. We rewrite the right-hand side as

$$(a-b)(\langle x_1, x_1\rangle + [x_2, x_2]) + (a+b)(\langle x_1, x_1\rangle - [x_2, x_2]) + \rho(x)$$
$$= \left(a - b + \frac{\rho(x)}{\|x\|^2}\right)\|x\|^2 + (a+b)E(x),$$

where $\|x\|^2 = \langle x_1, x_1\rangle + [x_2, x_2]$. Since this is visibly positive when $E(x)$ is positive and $x \neq 0$ is close to zero, we have proved part (b).

Note that part (b) of the Theorem is quite weak since unstable equilibrium points can "look" quite different. Could it be that the linearized system $x' = Ax$ has a saddle point at $x = 0$, but the non-linear system $x' = f(x)$ looks more like a source? A quite satisfactory answer is given in the following theorem, which is proved in Teschl's notes. This result is not included in the course, but it is good to know about it.

Hartman–Grobman Theorem: In the situation above, suppose that $\mathrm{Re}(\lambda) \neq 0$ for *all* eigenvalues of A. Then there exist open neighbourhoods U and V of 0, and a homeomorphism (that is, a continuous map with continuous inverse) $\phi : U \to V$, such that ϕ maps orbits of the system $x' = Ax$ to orbits of the system $x' = f(x)$, preserving direction of orbits.