Lyapunov's method

This topic is treated inadequatly in Teschl's book (in particular, the proof of Theorem 6.12 is incomplete). There is also a slight problem in the book of Andersson and Böiers. In these notes we summarize the main theory; examples and explanation are given in the lecture and textbooks.

We consider an autonomous system

$$x' = f(x) \tag{1}$$

defined in an open set $\Omega \subseteq \mathbb{R}^n$. We assume that f is locally Lipschitz continuous. We want to study stability of equilibrium points. Without loss of generality, we assume that 0 is an equilibrium point.

We will write $\dot{E} = \nabla E \cdot f$, so

$$\frac{d}{dt} E(x(t)) = \dot{E}(x(t))$$

when x is a solution of (1).

Definition: A continuosly differentiable function $E : \Omega \to \mathbb{R}$ is called a *Lyapunov function* if

- $E(x) \geq 0$, with equality only at $x = 0$,
- $\dot{E}(x) \leq 0$.

If, moreover, $\dot{E}(x) < 0$ for $x \neq 0$, E is called a *strict* Lyapunov function.

In particular, a Lyapunov function is decreasing along orbits.

Theorem: If a Lyapunov function exists, then 0 is stable.

Proof: Choose $\varepsilon > 0$ with $\{|x| < \varepsilon\} \subseteq \Omega$. Put $\gamma = \min_{|x|=\varepsilon} E(x) > 0$. Choose $\delta > 0$ so that $E(x) < \gamma$ when $|x| < \delta$ (possible by definition of continuity). In particular, if $|x(0)| < \delta$, then $E(x(0)) < \gamma$. Since E decreases along orbits, $E(x(t)) < \gamma$ for $t \geq 0$. Thus, $|x(t)| < \varepsilon$ for $t \geq 0$. This is the definition of stability.

Existence of a strict Lyapunov function implies asymptotic stability, but this is too weak for many applications. Therefore, we will prove the following stronger result.

Theorem: Suppose E is a Lyapunov function and that 0 is the only orbit contained in $\{x \in \Omega;\ \dot{E}(x) = 0\}$ (for instance, E could be strict). Then, 0 is asymptotically stable. Moreover, let Ω_c be the connected component of $\{x \in \Omega;\ E(x) < c\}$ that contains 0. Suppose $\overline{\Omega}_c$ is a compact subset of Ω. Then

$$x(0) \in \Omega_c \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0$$

(so Ω_c is a *domain of attraction* for 0).

It seems that the proof needs the following fact, glossed over in Andersson–Böiers.

Lemma: If f is locally Lipschitz continuous in an open set Ω (in the sense that it is Lipschitz continuous in an open neighbourhood of each point), then f is Lipschitz continuous on each compact set $K \subseteq \Omega$.

Proof: Suppose not. Then there is some compact K such that $|f(x) - f(y)|/|x-y|$ is unbounded on $\{x, y \in K;\ x \neq y\}$. So we can find sequences x_n, y_n in K with $|f(x_n) - f(y_n)|/|x_n - y_n| \to \infty$. By Bolzano–Weierstrass, we can assume that $x_n \to \xi$, $y_n \to \eta$. Clearly, $\xi = \eta$, since the limit would otherwise be $|f(\xi) - f(\eta)|/|\xi - \eta|$. Now take a neighbourhood U of ξ where the Lipschitz estimate

$$|f(x) - f(y)| \leq L|x-y|, \qquad x, y \in U$$

holds. For n big enough, x_n and y_n are in U, so that $|f(x_n) - f(y_n)|/|x_n - y_n| \leq L$; a contradiction.

Proof of Theorem: Take a solution with $x(0) \in \Omega_c$. Since $E(x(t))$ is decreasing, $x(t)$ stays in the compact set $\overline{\Omega}_c$. In particular (T Cor. 2.14, AB Sats 1.3), it is defined for all $t \geq 0$. By Bolzano–Weierstrass, we can choose a sequence $(t_k)_{k=1}^{\infty}$ of positive numbers with $t_k \to \infty$ and $x(t_k)$ convergent, say $x(t_k) \to \xi$ as $k \to \infty$. Let y denote the solution of

$$y' = f(y), \qquad y(0) = \xi, \tag{2}$$

and let $x_k(t) = x(t_k + t)$, so that x_k solves

$$x_k' = f(x_k), \qquad x_k(0) = x(t_k). \tag{3}$$

Since $x_k(t)$ and $y(t)$ belong to the compact set $\overline{\Omega}_c$, we have (Lemma above) a Lipschitz estimate

$$|f(x_k(t)) - f(y(t))| \leq L|x_k(t) - y(t)|, \qquad t \geq 0.$$

By the comparison theorem (T Thm. 2.8, AB Sats 1.8),

$$|x_k(t) - y(t)| \le |x(t_k) - \xi| e^{L|t|}, \qquad t \ge 0.$$

In particular, $\lim_{k\to\infty} x_k(t) = y(t)$, $t \ge 0$. Thus, $E(y(t)) = \lim_{k\to\infty} E(x(t + t_k))$. But since E is decreasing on orbits, this limit equals $\lim_{t\to\infty} E(t)$ and in particular does not depend on t. Thus, $E(y(t))$ is constant and $\dot{E}(y(t)) \equiv 0$. By our assumptions, this implies $y(t) \equiv 0$, which in turn implies $\lim_{t\to\infty} E(x(t)) = 0$ and $\lim_{t\to\infty} x(t) = 0$.

A variation of Lyapunov function can be used to prove instability.

Theorem: As above, consider the system (1) in an open neighbourhood Ω of the equilibrium point 0. Let $\Omega' \subseteq \Omega$ be an open set with $0 \in \overline{\Omega'}$. Suppose there is a continuously differentiable function E with

- $E > 0$ and $\dot{E} > 0$ in Ω',
- $E = 0$ on $\partial\Omega' \cap \Omega$.

Then, 0 is unstable.

Proof: Suppose $\{|x| \le \varepsilon\} \subseteq \Omega$. We will prove that any orbit starting in Ω' eventually satisfies $|x(t)| > \varepsilon$; this implies instability. Take such an orbit x and let $\rho = E(x(0)) > 0$. Put

$$K = \{x \in \Omega';\ |x| \le \varepsilon,\ E(x) \ge \rho\}.$$

This is a compact set, so $k = \min_K \dot{E}$ exists and is positive. As long as $x(t) \in K$, we have

$$\frac{d}{dt} E(x(t)) \ge k, \quad E(x(0)) = \rho.$$

This differential inequality implies $E(x(t)) \ge kt + \rho$ $(t \ge 0)$. But since E is bounded on K, we conclude that $x(t)$ must eventually leave K. Since E is strictly increasing along orbits, $x(t)$ leaves K at a point where $E(x) > \rho$, so it has to leave $\{|x| \le \varepsilon\}$.