Department of Mathematical Sciences, Chalmers& University of Gothenburg

MMA421, TMA013
Ordinary differential equations and dynamical systems
2010–03–11 kl. 8.30–13.30

You may not bring any notes, books or any other aids, not even a calculator!
To pass the exam (*i.e.* to obtain the grade "G" for (MMA421, GU), or grade "3" (TMA013, Chalmers)), you need 15 points. The final grade on the course depends also on the computer assignments.
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1. State and prove the Gronwall inequality (there is more than one version of the inequality, chose one). (5p)

2. Let $L : U \to \mathbb{R}$ be a Lyapunov function for the fixed point x_0 of a differential equation. Define S_δ as the connected component of $\{x \in U \big| L(x) \le \delta\}$ that contains. Prove that if S_δ is closed, it is a positively invariant set. Be careful to state all properties of a Lyapunov function that you use in the proof. Why is it important to mention "connected component", and why is it necessary that S_δ is closed? (5p)
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3. Compute the matrix exponential of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. (5p)

4. Consider the system
$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -x + x^2\,. \end{aligned}$$
   Sketch the phase portrait in as much detail as you can: find the fixed points, determine their character, separatrices if there are any etc. (5p)

5. Find the flow of the differential equation
$$\begin{aligned} \dot{x} &= -x^2\,, \\ \dot{y} &= -xy\,. \end{aligned}$$
   Be careful to state the domain of definition. (5p)

6. Consider the equation $\dot{x} = Ax$, where $x \in \mathbb{R}^n$, and A is a hyperbolic matrix. Let $z \in \mathbb{R}^n$, and let $\Omega_z = \mathbb{R}^n \setminus \{z\}$, *i.e.* $\mathbb{R}^n$ with the point z removed. If Ω_z is invariant under the flow, what is z? You must motivate your answer well. (5p)

Good luck!
Bernt W.