Department of Mathematical Sciences, Chalmers & University of Gothenburg

MMA421, TMA014
Ordinary differential equations and dynamical systems

2011–03–10 kl. 8.30–13.30

You may not bring any notes, books or any other aids, not even a calculator!
To pass the exam (*i.e.* to obtain the grade "G" for (MMA421, GU), or grade "3" (TMA013, Chalmers)), you need 15 points. The final grade on the course depends also on the computer assignments.
Information on when the exam has been graded, and information on when and where it can be inspected, will be given on the course web page. An e-mail will be sent when the results are registered in Ladok.
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1. For the following differential equations, find the fixed points and determine their stability.

   a)
$$\dot{x} = x + \frac{1}{x} - 4 \qquad \text{(here } x \in \mathbb{R}\text{)}.$$

   b)
$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin y \\ x + y \end{bmatrix} \qquad \text{(here } x, y \in \mathbb{R}\text{)}.$$

   (5p)

2. Compute the principal matrix solution $\Pi(t, t_0)$ for the system given by $A(t) = \begin{pmatrix} 2 & e^{-t} \\ 0 & 1 \end{pmatrix}$. (5p)

3. Consider the system

$$\begin{aligned} \dot{x} &= (x^2 + y^2)(2 + \cos(x^3 y^2)) \\ \dot{y} &= y \sin^2(x^2 + y^2) \end{aligned}$$

   Does it have any periodic solutions? Motivate your answer. (5p)

4. The Poincaré-Bendixon theorem states the following: *Let M is an open subset of $\mathbb{R}^2$, and $f \in C^1(M, \mathbb{R}^2)$, and consider the initial value problem $\dot{\phi} = f(\phi); \phi(0) = x$. Fix $x \in M$, and suppose that $\omega_{\pm}(x)$, the ω-limiting sets are non-empty and compact. Then one of the following holds:*

   a) $\omega_{\pm}(x)$ is a fixed orbit.

   b) $\omega_{\pm}(x)$ is a regular periodic orbit.

   c) $\omega_{\pm}(x)$ consists of (finitely many) fixed points $\{x_j\}$ and unique non-closed orbits $\gamma(y)$ such that $\omega_{\pm}(y) \in \{x_j\}$.

   Sketch three different phase-portraits that illustrate the three cases. (5p)

5. Consider a linear system $\dot{x} = Ax$, where A is a real $n \times n$-matrix. State and prove a theorem which states how the stability of the fixed point $x = 0$ can be determined from knowledge of the eigenvalues (and eigenvectors) of A. (5p)

6. State the definition of a Liapunov function. Let $L = U \to \mathbb{R}$ be a Liapunov function for the fixed point $x_0 \in U$, and let S_δ be the connected component of the set $\{x \in U \text{ such that } L(x) \leq \delta\}$. Let $B_\varepsilon(x_0) = \{x \text{ such that } |x - x_0| < \varepsilon\}$ be the open ball of radius ε around x_0. Prove that for every $\varepsilon > 0$, there is $\delta > 0$ such that $S_\delta \subset B_\varepsilon(x_0)$. (5p)

Good luck!
Bernt W.