Department of Mathematical Sciences, Chalmers& University of Gothenburg

MMA421, TMA014
Ordinary differential equations and dynamical systems

2011–08–27 kl. 8.30–13.30

You may not bring any notes, books or any other aids, not even a calculator!
To pass the exam (*i.e.* to obtain the grade "G" for (MMA421, GU), or grade "3" (TMA014, Chalmers)), you need 15 points. The final grade on the course depends also on the computer assignments.
Telephone: Magnus Önnheim, tel. 0703-088304

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1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous, the the initial value problem

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

has a unique solution for t belonging to a sufficiently small interval around 0. (5p)

2. State and prove the Gronwall inequality (there are several versions, choose one). (5p)

3. Show that $L(x, y, z) = x^2 - \cos y + z^4$ is a Lyapunov function at the equilibrium point $(0, 0, 0)$ of the system

$$\begin{aligned} \dot{x} &= -2z^3 - z\sin y \\ \dot{y} &= 2xz - \sin y \\ \dot{z} &= x - \sin^3 . \end{aligned}$$

What conclusion can you draw about the equilibrium point? Why couldn't linearization give the same information? (5p)

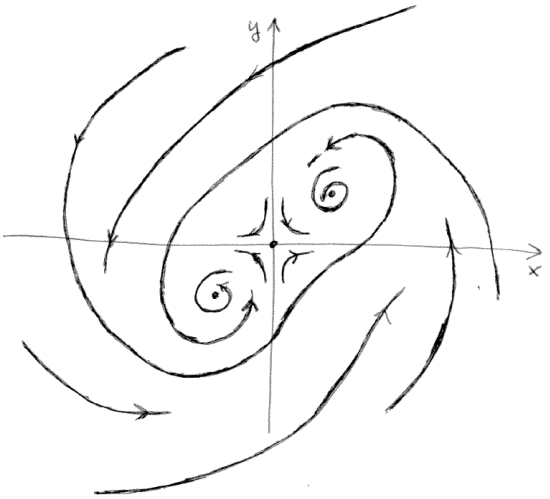
4. Compute the matrix exponential of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. (5p)

5. Find the flow of the system

$$\begin{aligned} \dot{x} &= 1/\sqrt{y} \\ \dot{y} &= -1/\sqrt{x} . \end{aligned}$$

Be careful to state the domain of definition. It is possible to give the solution completely explicitly, but I give full marks for an implicitly given solution (*i.e.* an equation involving x,y and t but not $\dot{x}$ or $\dot{y}$.) (5p)

6. Give an example of a vector field (the right hand side of a system of odes) whos phase portrait is similar to figure 1. Prove that your example has has the right properties. (5p)



Figur 1: Phase portrait for problem 6

Good luck!
Bernt W.