Department of Mathematical Sciences, Chalmers & University of Gothenburg

MMA421, TMA014
Ordinary differential equations and dynamical systems

2012–03–08 kl. 8.30–12.30

You may not bring any notes, books or any other aids, not even a calculator!
To pass the exam (*i.e.* to obtain the grade "G" for (MMA421, GU), or grade "3" (TMA014, Chalmers)), you need 15 points. The final grade on the course depends also on the computer assignments.
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1. Let C be a closed subset of a Banach space X, and let $K : C \to C$ be a contraction. Explain what is meant by the word *contraction*, and state and prove the *contraction mapping principle*. (5p)

2. State the definition of a Liapunov function. Let $L = U \to \mathbb{R}$ be a Liapunov function for the fixed point $x_0 \in U$, and let S_δ be the connected component of the set $\{x \in U \text{ such that } L(x) \leq \delta\}$. Let $B_\varepsilon(x_0) = \{x \text{ such that } |x - x_0| < \varepsilon\}$ be the open ball of radius ε around x_0. Prove that for every $\varepsilon > 0$, there is $\delta > 0$ such that $S_\delta \subset B_\varepsilon(x_0)$. (5p)

3. Use Liapunov's method for proving that the origin is asymptotically stable for the following system of ode's:

$$\begin{aligned} \dot{x} &= (x-y)(x^2+y^2-1) \\ \dot{y} &= (x+y)(x^2+y^2-1) \end{aligned}$$

   Be careful to show that your Liapunov function satisfies all requirements of a Liapunov function. There may be other methods for proving stability, but the one that I ask for is Liapunov's method. (5p)

4. Sketch the phase portrait of the system

$$\begin{aligned} \dot{x} &= x(2-y) \\ \dot{y} &= 2x^2 - y \end{aligned}$$

   (5p)

5. Consider the linear system $\dot{x} = Ax$, where $x(t) \in \mathbb{R}^3$ and A is a real 3×3-marix. Determine the flow generated by this differential equation, and determine all possible ω_+ limiting sets (depending on the matrix A and initial points $x(0)$). (5p)

6. Find a local change of coordinates that "straightens out" the vector field

$$F = \begin{pmatrix} F_1(x,y) \\ F_2(x,y) \end{pmatrix} = \begin{pmatrix} x + y^2 \\ -y \end{pmatrix}$$

   near the point $(0,1)$, *i.e.* a transformation such that in the new coordinates, the vector field is given by $\tilde{F} = (0,1)$.

   (Hint: begin by computing the flow of the ode for the original vector field; doing this correctly gives 2 points).

   Is it possible to find a global change of coordinates with this property? (5p)

Good luck!
Bernt W.