Department of Mathematical Sciences, Chalmers & University of Gothenburg

MMA421, TMA014
Ordinary differential equations and dynamical systems
2012–06–05 kl. 8.30–12.30

You may not bring any notes, books or any other aids, not even a calculator!
To pass the exam (*i.e.* to obtain the grade "G" for (MMA421, GU), or grade "3" (TMA014, Chalmers)), you need 15 points. The final grade on the course depends also on the computer assignments.
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1. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous, the the initial value problem

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution for t belonging to a sufficiently small interval around 0. (5p)

2.
   - Explain what is meant by the *Jordan canonical form* of a matrix, and state a theorem concering the existence of such forms.
   - What does the Jordan form of the matrix for a linear system of differential equation tell you about the solutions the system? (5p)

3. Compute the matrix exponential of $A = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$.

4. Sketch the phase portrait of the system

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - \frac{x}{M} - y\right), \\ \frac{dy}{dt} &= ky(x-1). \end{aligned}$$

You may consider the case $M = 5$, $k = 1$. (5p)

5. Show that if the origin is asymptotically stable for one of the following two systems of ode's, then the the same holds for the other one:

$$\begin{aligned} 1) \qquad \dot{x} &= f(x) \\ 2) \qquad \dot{x} &= f(x)h(x) \end{aligned}$$

Here $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\quad h(0) = 0$ and $h \in C^1(\mathbb{R}^n, \mathbb{R})$, $h(0) > 0$. (5p)

6. Consider the differential equation

$$\begin{aligned} (1-t^2)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n+1)y &= 0, \\ y(0) &= 1, \\ y'(0) &= 0. \end{aligned}$$

This is Legendre's differential equation of order n. Determine an interval $[0, t_0]$ such that the basic existence theorem for ode's guarantees that a solution exists in this interval. (5p)

Good luck!
Bernt W.