ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

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1. Generalities on dynamical systems

Very broadly speaking, a **dynamical system** is nothing but a group action, that is to say, a group G and a set X, together with an **action map** $\alpha : G \times X \to X$ such that

$$\alpha(st, x) = \alpha(s, \alpha(t, x)), \quad \text{for all } s, t \in G \text{ and } x \in X.$$

It is often convenient to abbreviate this notation, and simply write $t.x := \alpha(t, x)$. Given a point $x \in X$, we call the subset $G.x = \alpha(G, x) \subset X$ the **G-orbit of** x. The research area of dynamical systems is concerned with the question how the various G-orbits are "distributed" in X. We say that a point $x \in X$ is

- **fixed** if $t.x = x$ for all $t \in G$, in which case $G.x = x$.
- **periodic** if there is a finite-index subgroup $G_o < G$ such that x is fixed by G_o, in other words, if $G.x$ is a finite set.

The question of "distribution" of points only gets interesting if we require G and X to have additional structures. In this course, we will consider the case when both G and X are topological, and the action map $G \times X \to X$ is continuous. To avoid trivialities, we shall further assume that G is non-compact; we then say that a sequence (g_n) in G **tends to infinity** if for every compact set $K \subset G$, there is an index n_K such that $g_n \notin K$ for all $n > n_K$. We further say that a point $x \in X$ is

- **recurrent** if there exists a sequence (g_n) in G which tends to infinity such that $g_n.x \to x$.
- **transitive** if $G.x$ is dense in X.

In this course, our main focus will be on the groups $G = (\mathbb{R}, +)$ and $G = (\mathbb{Z}, +)$. In the second case, the action map α is completely determined by the map $T : X \to X$ given by $T(x) = \alpha(1, x)$. In particular, a point is periodic if there exists $p \geq 1$ (called the periodic of x) such that $T^p x = x$ (and fixed if it is periodic with $p = 1$). In the case when $G = (\mathbb{R}, +)$, we shall write $T_t(x) = \alpha(t, x)$, and refer to the action as a **flow**. If one fleshes out the definitions above in this case, we see that a point $x \in X$ is

- **fixed** if $T_t(x) = x$ for all $t \in \mathbb{R}$.
- **periodic** if there exists $t_o > 0$ (called the **period of** x) such that $T_{t_o}(x) = x$, in which case we have $T_{t+t_o}(x) = T_t(x)$ for all $t \in \mathbb{R}$.
- **recurrent** if there exists (t_j) with $t_j \to \infty$ such that $T_{t_j}(x) \to x$.
- **transitive** if for every $y \in X$, there is a sequence (t_j) such that $T_{t_j}(x) \to y$.

In the first part of the course, we will address the following questions:

(i) Why do we study flows?
(ii) How do we construct (interesting) flows?
(iii) When are there fixed/periodic/recurrent/transitive points for flows?

Once we have produced some classes of "interesting" flows, it is natural to ask how they behave under "perturbations" - indeed, in most realistic scenarios, the knowledge of the flow can only be assumed to be approximate, why one is lead to study all flows "close" to the one at hand. In general this is an impossible task, but for a special class of flows satisfying enough "hyperbolicity", it turns out that all small perturbations are merely coordinate changes, and behave, on the whole, exactly as the original flow.

1.1. **Motivation through Hamiltonian mechanics**

Consider a "classical system" of N particles with positions $q^{(i)} = (q_1^{(i)}, q_2^{(i)}, q_3^{(i)})$ and momenta $p^{(i)} = (p_1^{(1)}, p_2^{(2)}, p_3^{(3)})$, $i = 1, \ldots, N$. These are coordinates in a $6N$-dimensional "phase space" R^{6N}. The motion of the particles are governed by **Hamilton's equations** (Newton's laws in disguise),

$$\frac{dq_k^{(i)}}{dt} = \frac{\partial H}{\partial p_k^{(i)}} \quad \text{and} \quad \frac{dp_k^{(i)}}{dt} = -\frac{\partial H}{\partial q_k^{(i)}}, \tag{1.1}$$

where H denotes the **Hamiltonian**, given by

$$H(q,p) = \sum_{i=1}^{N} \frac{|p^{(i)}|^2}{2m_i} + U(q^{(1)}, \ldots, q^{(N)}),$$

for particle masses (m_i) and a **potential energy** U. Under some natural assumptions on U, we shall prove in Section 4 below that the solutions $t \mapsto (q(t), p(t))$ to Hamilton's equations exist for all $t \in \mathbb{R}$ and for all initial values $x_o \in \mathbb{R}^{6N}$, so we get a map $(t, x_o) \mapsto T_t(x_o) = (q(t), p(t))$, where $q = (q^{(1)}, \ldots, q^{(N)})$ and $p = (p^{(1)}, \ldots, p^{(N)})$ solve (1.1) with $x_o = (q(0), p(0))$. We shall later show that this map is indeed a flow. It is a special feature of such "Hamiltonian flows" that if $H(x_o) = E$ for some real number E, known as the **total energy** of the system, then $H(T_t(x_o)) = E$ for all t; in other words, the flow preserves the closed subset

$$M = \left\{(q,p) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N} : H(q,p) = E\right\} \subset \mathbb{R}^{6N}.$$

After we have discussed fixed points/periodic orbits for Hamiltonian flows, we shall prove the famous **Poincaré Theorem**, which roughly asserts that there is an "abundance" of recurrent points for these flows; this gives rise to the following counterintuitive observation. If one picks any point x_o with $H(x_o) = E$, then there will be recurrent points for the flow which are arbitrarily close to x_o. In particular, one can imagine a gas with $N \approx 10^{24}$ particles at time $t = 0$, with positions $q(0)$ very close to each other, but with high momenta $p(0)$ so that $H(q(0), p(0)) = E$. This is clearly a very strange situation for a gas (in a closed room, this would mean that except for a small corner of the room, there would be vacuum), but Poincaré's Theorem tells us that if we let these particles evolve in time, and wait long enough, then there will be occasions when the gas returns to essentially the same state - all concentrated in a small corner of the room.

In order for us to arrive at Poincare's Theorem, we must first go through quite a lot of technicalities surrounding differential equations. In general, we study differential equations of the form

$$\dot{x} = F(x), \quad x(0) = x_o, \tag{1.2}$$

where F is some $\mathbb{R}^M$-valued "nice" function on an open subset $U \subset \mathbb{R}^M$ (these are often called **vector fields**), and x_o is a point in U. The Hamiltonian flows correspond to $M = 6N$, $U = \mathbb{R}^M$ and F is given by (1.1). We shall show that there is always a "local flow" $(t, x_o) \mapsto T_t(x_o) = x(t)$, where x is the *unique* solution to (1.2), defined on some interval $J \subset \mathbb{R}$ which contains zero. Under some additional assumptions, this solution can be extended to the whole real line, in which case $(t, x) \mapsto T_t(x)$ is a *bona fide* flow. Furthermore, for every fixed t, the Jacobian $DT_t(x_o)$ exists; for Hamiltonian flows, we shall show that its determinant equals one for all t - this property is called **measure-preservation** - and once this has been established, Poincaré's Theorem is not far away.

1.2. **Special flows**

Although flows coming from differential equations are certainly interesting, they are rather technical to deal with. It is therefore useful to have "toy examples" to enhance the intuition of what one can expect from flows in general. In this subsection, we shall define a class of such "toy examples" known as **special flows**; they have the important characteristic of being determined by a $(\mathbb{Z},+)$-action $S : Y \to Y$, where Y is compact, and a continuous "roof" function $r : Y \to (0,\infty)$. We set $r_o \equiv 0$,

$$r_n(y) = \sum_{k=0}^{n-1} r(S^k(y)), \quad \text{for } n \geq 1,$$

and define r_n for all n in such a way that $r_{m+n}(x) = r_m(x) + r_n(S^m(x))$ for all m, n. The special flow $T_\bullet$ determined by (S, r) is defined on the set

$$X = \{(y,u) : y \in Y,\ 0 \leq u \leq r(y)\}/\sim,$$

where $(y,0) \sim (S(y), r(y))$. With respect to the quotient topology, X is compact, and the map $T : \mathbb{R} \times X \to X$ defined by

$$T_t(y,u) = (S^n(y), t + u - r_{n-1}(y)), \quad \text{if } r_{n-1}(y) \leq t + u < r_n(y) \tag{1.3}$$

is a continuous flow on X (please check all of this yourself!).

There is a very strong dynamical correspondence between $S \curvearrowright Y$ and $T_\bullet \curvearrowright X$, as the following three lemmas show.

Lemma 1.1. *If $S^n(y) = y$ and $0 \leq u < r(y)$, then $T_{t_o}(y,u) = (y,u)$ for $t_o = r_{n-1}(y)$. In particular, if S has periodic points, so does the flow $T_\bullet$.*

Proof. Since $r > 0$, we have $r_{n-1}(y) \leq t_o + u < r_n(y) = r_{n-1}(y) + r(S^n(y))$ for all $0 \leq u < r(y)$, and thus $T_{t_o}(y,u) = (S^n(y), t_o + u - r_{n-1}(y)) = (y,u)$. □

Lemma 1.2. *If $S^{n_k}(y) \to y$ and $0 \leq u < r(y)$, there is a sequence (t_k) which tends to infinity such that $T_{t_k}(y,u) \to (y,u)$. In particular, if S has recurrent points, so does the flow $T_\bullet$.*

Proof. Since r is continuous and $S^{n_k}(y) \to y$, there is a sequence $u \geq \varepsilon_k \searrow 0$ such that $u < r(S^{n_k}(y)) + \varepsilon_k$ for all (sufficiently large) k; if $u = 0$, we set $\varepsilon_k = 0$ for all k. Let $t_k = r_{n_k - 1}(y) - \varepsilon_k$, and note that

$$r_{n_k-1}(y) \leq t_k + u < r_{n_k}(y) + r_{n_k-1}(y) + r(S^{n_k}(y)).$$

Hence, $T_{t_k}(y,u) = (S^{n_k}(y), u - \varepsilon_k) \to (y,u)$. □

Along the same lines, we can prove:

Lemma 1.3. *Suppose that $y, z \in Y$ and $S^{n_k}(y) \to z$. Then, for any $0 \leq u < r(y)$ and $0 \leq v < r(z)$, there is a sequence (t_k) such that $T_{t_k}(y,u) \to (z,v)$. In particular, if S has transitive points, so does the flow $T_\bullet$.*

Hence, in order to answer questions about the existence of periodic/recurrent/transitive points for special flows, we only need to produce such points for the corresponding $\mathbb{Z}$-action S. Let us consider an example.

Example 1.4 (Recurrent/Transitive points for S)**.** Let $Y = \mathbb{R}/\mathbb{Z}$, which we can think of as the compact interval $[0,1]$ with the endpoints identified. Fix an irrational α, and define $S : Y \to Y$ by $S(y) = y + \alpha$ modulo 1. First note that S does not have any periodic points; indeed, if $S^p(y) = y$, then $p\alpha = 0$ modulo 1, which means that $\alpha = m/p$ for some integer m, contradicting the irrationality of α. By Lemma 1.1, no special flow $T_\bullet$ associated to S can have periodic points.

We claim that every point is recurrent. Since $S^n(y) = y + n\alpha = y + S^n(0)$ it suffices to show that 0 is recurrent. Let $N \geq 1$ be an integer, and partition Y into intervals of the form $[k/N, (k+1)/N)$

for $k = 0,\dots,N-1$. Since α is irrational, the orbit $\{n\alpha \mod 1\}$ does not hit the endpoints of these intervals. Hence, if we consider the string $n\alpha \mod 1$, $n = 1,\dots,N+1$, then at least two distinct points, say $m_N\alpha$ and $n_N\alpha$ must fall into the same interval, and thus $(m_N - n_N)\alpha \mod 1$ is $1/N$-away from 0. Since N can be chosen arbitrarily large, we see that 0 is recurrent. By Lemma 1.2, any special flow associated to S has recurrent points.

Finally, we leave it as an exercise to show that every point in Y is actually transitive.

2. Preliminaries and notation

3. Picard-Lindelöf's Theorem and some of its consequences

Let V be a finite-dimensional real vector space, and $U \subset V$ a non-empty open set. Given a point $x_o \in U$ and a continuous function $F : U \to V$, the associated **initial value problem (IVP)** asks for an open interval $I \subset \mathbb{R}$ around zero, and $\phi_{x_o} \in C^1(I,U)$ with $\phi_{x_o}(0) = x_o$ such that $\dot{\phi}_{x_o} = F \circ \phi_{x_o}$. In this section, we address the following three natural questions related to this problem:

(i) Can the IVP always be solved?
(ii) If a solution exists, is it unique?
(iii) If solutions always exist and are unique, what can we say about the map $(t,x_o) \mapsto \phi_{x_o}(t)$?

3.1. Initial value problems viewed as fixed points

Fix a norm $|\cdot|_V$ on V. For $x_o \in V$ and $\delta > 0$, we denote by $\overline{B}_\delta(x_o)$ the closed ball with radius δ around x_o for the metric $d_V(u,v) = |u-v|_V$ on V, and for $r > 0$, we set

$$\mathcal{D}_{\delta,r}(x_o) = \{\phi \in C\big([-r,r],\overline{B}_\delta(x_o)\big) : \phi(0) = x_o\}.$$

By Lemma A.1, $C\big([-r,r],\overline{B}_\delta(x_o)\big)$ is a complete metric space with respect to

$$\rho(\phi,\psi) = \sup_{|s|\le r} |\phi(s) - \psi(s)|_V, \quad \phi,\psi \in C\big([-r,r],\overline{B}_\delta(x_o)\big),$$

and so is $\mathcal{D}_{\delta,r}(x_o)$, being a closed subset of $C\big([-r,r],\overline{B}_\delta(x_o)\big)$.

Let us fix $x_o \in V$ and $\delta > 0$, and assume that $F : \overline{B}_\delta(x_o) \to V$ is continuous. Given $r > 0$, we define the map $K_{x_o} : C\big([-r,r],V\big) \to C\big([-r,r],V\big)$ by

$$(K_{x_o}\phi)(t) = \begin{cases} x_o + \int_0^t (F\circ\phi)(s)\,ds & 0 \le t \le r \\ x_o - \int_t^0 (F\circ\phi)(s)\,ds, & -r \le t \le 0 \end{cases}, \tag{3.1}$$

and note that if $\phi \in C\big([-r,r],V\big)$ is a fixed point of K_{x_o}, then $\phi(0) = x_o$ and ϕ must be differentiable on $(-r,r)$, where it satisfies $\dot{\phi} = F\circ\phi$. In other words, ϕ is a (local) solution to the IVP. Conversely, if ϕ is a solution to the IVP above, then its restriction to $[-r,r]$ is a fixed point for K_{x_o}, provided of course that $[-r,r] \subset I$. Our first goal is to impose the right conditions on $r > 0$ and F to ensure that K_{x_o} has a fixed point.

Since F is assumed continuous, $\|F\|_\infty$ is finite, so if we choose $r > 0$ such that $r\|F\|\infty \le \delta$, then

$$|(K_{x_o}\phi)(t) - x_o|_V \le \int_0^t |(F\circ\phi)(s)|_V\,ds \le r\|F\|_\infty, \quad \text{for all } \phi \in \mathcal{D}_{\delta,r}(x_o),$$

for all $0 \le t \le r$ (and similarly for $-r \le t \le 0$) and thus $K_{x_o}(\mathcal{D}_{\delta,r}(x_o)) \subset \mathcal{D}_{\delta,r}(x_o)$. Let us fix a non-empty closed subset $\mathcal{C} \subset \mathcal{D}_{\delta,r}(x_o)$ with $K_{x_o}(\mathcal{C}) \subset \mathcal{C}$, and suppose that there is $L > 0$ such that

$$\|F\circ\phi - F\circ\psi\|_\infty \le L\|\phi-\psi\|_\infty, \quad \text{for all } \phi,\psi \in \mathcal{C}. \tag{3.2}$$

Then, for all $0 \le t \le r$,

$$|(K_{x_o}\phi)(t) - (K_{x_o}\psi)(t)|_V \le \int_0^t |(F\circ\phi)(s) - (F\circ\psi)(s)|_V\,ds \le rL\|\phi-\psi\|_\infty,$$

and similarly for $-r \le t \le 0$. In particular, if $rL < 1$, then $K_{x_o} : \mathscr{C} \to \mathscr{C}$ is a strict contraction. Since $\mathscr{C}$ is a closed subset of the complete metric space $\mathscr{D}_{\delta,r}(x_o)$, it too must be complete, and thus Theorem A.2 tells us that K_{x_o} has a unique fixed point $\phi \in \mathscr{C}$. We summarize this discussion in the following lemma.

Lemma 3.1. *Let $F : \overline{B}_\delta(x_o) \to V$ be a continuous function, and pick $r > 0$ such that $r\|F\|_\infty \le \delta$. Then, for every $\mathscr{C} \subset \mathscr{D}_{\delta,r}(x_o)$ with $K_{x_o}(\mathscr{C}) \subset \mathscr{C}$ such that* (3.2) *holds for some L with $rL < 1$, there exists a unique $\phi \in \mathscr{C}$ such that $\dot{\phi} = F \circ \phi$ on $(-r,r)$.*

Corollary 3.2. *Suppose that $F : \overline{B}_\delta(x_o) \to V$ is L-Lipschitz. Then, for any $r < \min\big(\delta/\|F\|_\infty, 1/L\big)$, there exists a unique $\phi : [-r,r] \to \overline{B}_\delta(x_o)$ with $\phi(0) = x_o$, which is differentiable on $(-r,r)$, and where it satisfies $\dot{\phi} = F \circ \phi$.*

Proof. One readily checks that (3.2) holds for $\mathscr{C} = \mathscr{D}_{\delta,r}(x_o)$ for any $r > 0$. By choosing r as in the corollary, the conditions in Lemma 3.1 are satisfied. □

3.2. Linear initial value problems

Let W be a finite-dimensional real vector space, $J \subset \mathbb{R}$ a compact interval around zero and $A : J \to \mathrm{End}(W)$ a continuous map. Our aim here is to show that there exists a unique $\Phi : J^o \to \mathrm{End}(W)$ with $\Phi(0) = \mathrm{Id}_W$ which is differentiable and satisfies $\dot{\Phi} = A\Phi$.

As a first step, we reformulate this variant of an initial value problem in a language which better aligns with the one used in the previous subsection. Fix a norm $|\cdot|_{\mathrm{End}(W)}$, and define the norm $|\cdot|_V$ on the real vector space $V = \mathbb{R} \times \mathrm{End}(W)$ by $|\cdot|_V = \max(|\cdot|, |\cdot|_{\mathrm{End}(W)})$.

For any $\delta > 0$, we have $\overline{B}_\delta(0, \mathrm{Id}_W) = [-r,r] \times \overline{B}_\delta(\mathrm{Id}_W)$. If $\delta > 0$ is chosen so that $[-\delta,\delta] \subset I$, we can define the continuous map $F : \overline{B}_\delta(0, I_W) \to V$ by

$$F(t,\Delta) = (1, A(t)\Delta), \quad \text{for } |t| \le \delta \text{ and } |\Delta - \mathrm{Id}_W| \le \delta.$$

and for any $0 < r \le \delta$, we set

$$\mathscr{C} = \big\{\phi : [-r,r] \to [-r,r] \times \mathrm{End}(W) : \phi(t) = (t, \Phi(t)),\ \ \Phi \in C\big([-r,r], \overline{B}_\delta(\mathrm{Id}_W)\big) \ \text{ and } \ \Phi(0) = \mathrm{Id}_W\big\}.$$

We note that $\mathscr{C}$ is a closed subset of $\mathscr{D}_{\delta,r}(0, \mathrm{Id}_W)$, and we claim that if

$$r \max\big(1, \|A\|_\infty(|\mathrm{Id}_W|_{\mathrm{End}(W)} + \delta)\big) \le \delta \quad \text{and} \quad r\|A\|_\infty < 1$$

then $r\|F\|_\infty \le \delta$, (3.2) holds for some L with $rL < 1$ and $K_{(0,\mathrm{Id}_W)}$ maps $\mathscr{C}$ into itself. Hence, by Lemma 3.1, there is a unique $\phi : [-r,r] \to V$ such that $\dot{\phi} = F \circ \phi$ on $(-r,r)$; fleshing this out yields a continuous map $\Phi : [-r,r] \to \mathrm{End}(W)$ with $\Phi(0) = \mathrm{Id}_W$ such that Φ is differentiable on $(-r,r)$ and satisfies $\dot{\Phi} = A\Phi$.

Let us now try to solve the same differential equation, but for a different initial value Δ_o in $\mathrm{End}(W)$. With the same Φ as above, one readily verifies that $\Psi(t) = \Phi(t)\Delta_o$ satisfies

$$\dot{\Psi} = A\Psi, \quad \Psi(0) = \Delta_o.$$

Theorem 3.3.

Theorem 3.4. *Let $I \subset \mathbb{R}$ be compact interval and suppose that $A : I \to \mathrm{End}(V)$ is continuous. Then there exists a* unique $\Phi \in \mathscr{C}^1(I^o, \mathrm{End}(V))$ *such that*

$$\dot{\Phi} = A\,\Phi, \quad \Phi(0) = \mathrm{Id}_V\,. \tag{3.3}$$

Proof. □

Theorem 3.5.

$$\det \Phi(t) = \exp\Big(\int_0^t \mathrm{tr}\, A(s)\, ds\Big)$$

Theorem 3.6.

$$\dot{\Delta}_x = DF(\phi_x)\Delta_x, \quad \Delta_x(0) = \mathrm{Id}_V \tag{3.4}$$

Theorem 3.7.

4. Hamiltonian flows

Appendix A. Complete metric spaces

Let X be a set. A function $d : X^2 \to [0,\infty)$ is called a **metric** if

$$d(x,x) = 0 \quad \text{and} \quad d(x,y) = d(y,x), \quad \text{for all } x,y \in X,$$

and

$$d(x,y) \leq d(x,z) + d(z,y), \quad \text{for all } x,y,z \in X. \tag{A.1}$$

If d is a metric on X, we refer to (X,d) as a **metric space**. If $x \in X$ and $r > 0$, then the open and closed **balls of radius** r **around** x are defined as the sets

$$B_r(x) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad \overline{B}_r(x) = \{y \in X : d(x,y) \leq r\}$$

respectively. Any metric space (X,d) can be endowed with a topology by declaring a a non-empty set $U \subset X$ to be **open** if for every $x \in U$, there is $r > 0$ such that $\overline{B}_r(x) \subset U$; this is the **metric topology**. In particular, a sequence (x_n) in X converges to a point x in the metric topology if and only if $d(x,x_n) \to 0$. We say that a sequence (x_n) in X is **Cauchy** if for every $\varepsilon > 0$, there is an integer N_ε such that

$$d(x_m,x_n) < \varepsilon, \quad \text{for all } m,n \geq N_\varepsilon, \tag{A.2}$$

and a metric space (X,d) is **complete** if every Cauchy sequence in X converges. Finally, we say that (X,d) is (sequentially) **compact** if every sequence in X has a convergent sub-sequence.

A.1. Spaces of maps

Suppose (X,d_X) and (Y,d_Y) are metric spaces, endowed with their respective metric topologies. The set $C_b(X,Y)$ of bounded continuous maps from X into Y can be equipped with the metric

$$\rho_{X,Y}(\phi,\psi) = \sup_{x\in X} d_Y(\phi(x),\psi(y)), \quad \text{for } \phi,\psi \in C_b(X,Y). \tag{A.3}$$

Note that if (X,d_X) is compact, then every continuous map from X into Y is automatically bounded, in which case we simply write $C(X,Y)$ instead of $C_b(X,Y)$. We shall prove:

Lemma A.1. *If (X,d_X) is a compact and (Y,d_X) complete, then $(C(X,Y),\rho_{X,Y})$ is complete.*

Proof. Let (ϕ_n) be a Cauchy sequence in $C(X,Y)$. Then, for any $x \in X$, the sequence $y_n = f_n(x)$ is a Cauchy sequence in Y, and thus converges to a point in Y which we denote by $f(x)$. We need to show that for the function $f : X \to Y$ so obtained, $\rho_{X,Y}(f,f_n) \to 0$; since each f_n is continuous, this readily implies that f is continuous as well.

We argue by contradiction, and assume that there is a sequence (x_n) in X such that

$$d_Y(f(x_n),f_n(x_n) \not\to 0.$$

By the definition of f, we can find (m_n) such that $d_Y(f(x_n),f_{m_n}(x_n)) < 1/n$ for all n, and thus

$$\begin{aligned} d_Y(f(x_n),f_n(x_n)) &\leq d_Y(f(x_n),f_{m_n}(x_n)) + d_Y(f_{m_n}(x_n),f_n(x_n)) \\ &\leq d_Y(f(x_n),f_{m_n}(x_n)) + \rho(f_{m_n},f_n) \\ &< \frac{1}{n} + \rho(f_{m_n},f_n), \end{aligned}$$

for all n. Since (f_n) is Cauchy in $C(X,Y)$, the right hand side must tend to zero, which in turn implies that $d_Y(f(x_n),f_n(x_n)) \to 0$, contradicting our assumption. □

A.2. **Contractions and fixed points**

If (X,d_X) and (Y,d_Y) are metric spaces, and $L>0$, we say that a map $F:X\to Y$ is **L-Lipschitz** if

$$d_Y(F(x),F(x')) \le L\, d_X(x,x'), \quad \text{for all } x,x' \in X.$$

If $L\le 1$ and $(X,d_X)=(Y,d_Y)$, then F is called a **L-contraction**, and it is called a **strict contraction** if $L<1$.

Theorem A.2 (Banach's Fixed Point Theorem)**.** *Every strict contraction on a complete metric space has a unique fixed point.*

Proof. Let (X,d) be a complete metric space and $K:X\to X$ a L-contraction with $L<1$. Pick $x_o\in X$ and set $r=d(x_o,K(x_o))$. Define $x_n=K^n(x_o)$ for $n\ge 1$, and note that for all $m<n$, we have

$$\begin{aligned} d(x_m,x_n) &\le L^m d(x_o,x_{n-m}) \le L^m\Big(\sum_{k=0}^{n-m-1} d(x_k,x_{k+1})\Big) \\ &\le L^m\Big(\sum_{k=0}^{\infty} L^k r\Big) = \frac{L^m}{1-L} r, \end{aligned}$$

and the right-hand side can be made arbitrary small by taking m large. Hence, the sequence (x_n) is Cauchy in X, and thus converges to a point x. Since $K:X\to X$ is continuous,

$$K(x) = \lim_n K(x_n) = \lim_n x_{n+1} = x,$$

and we see that x is a fixed point for K. Finally, if there would be another fixed point y in X, distinct from x, then $0<d(x,y)=d(K(x),K(y))<Ld(x,y)$, which contradicts $L<1$. ☐