Diffeomorphisms

Discrete dynamical systems: $x_{n+1} = f(x_n)$ $\qquad f: \mathbb{R}^n \to \mathbb{R}^n$

$f: M \to M$

Def f is a C^r-map if it is r times continuously differentiable

f is "smooth" if it is a C^∞-map

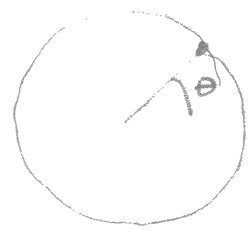
f is a (C^k-) diffeomorphism if it is a bijection and f and f^{-1} are C^1 (C^k)

f is a homeomorphism if it is a bijection and f and f^{-1} are C^0.

Diffeomorphisms of the circle

Ex $\quad R_\alpha: \theta \mapsto \theta + \alpha \pmod 1$

(or mod 2π)



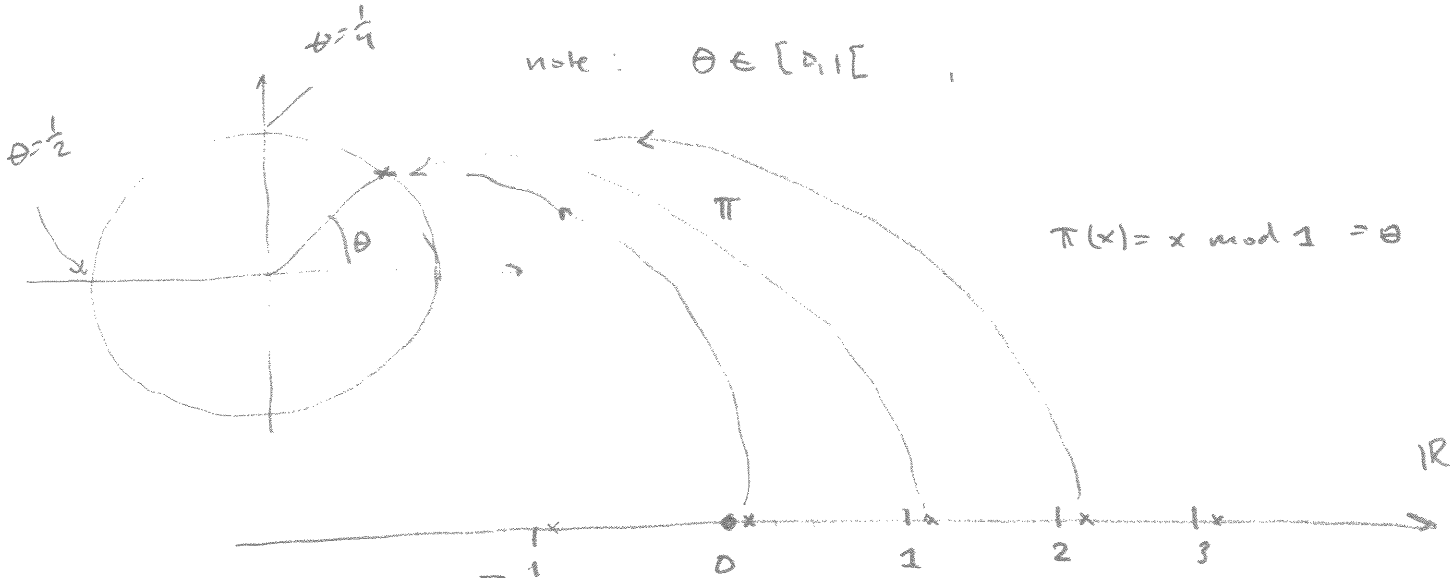
Def Let $f: S^1 \to S^1$ be a homeomorphism.

Suppose that there is a map $\bar{f}: \mathbb{R} \to \mathbb{R}$ such that

$\pi(\bar{f}(x)) = f(\pi(x))$ where $\pi(x) = x \bmod 1$ $(= \theta)$

Then $\bar{f}$ is called a lift of f onto $\mathbb{R}$

note: $\theta \in [0,1[$,



$\pi(x) = x \bmod 1 = \theta$

$$\begin{tikzcd} x & \bar f(x) \\ \mathbb{R} \arrow[r, "\bar f"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "f"] & S^1 \\ \pi(x) & \pi(\bar f(x)) \end{tikzcd}$$

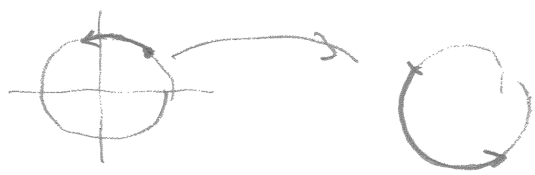
If $f(\pi(x)) = \pi(\bar f(x))$ for all x

we say that the diagram "commutes"

Dynamical systems on M is more complicated if $M \neq \mathbb{R}^n$. But lifts can help, when they exist.

Prop If $\bar f$ is a lift of an orientation preserving map $f : S^1 \to S^1$

Then $\bar f(x+1) = \bar f(x) + 1$ for every $x \in \mathbb{R}$.



Proof $\pi(x) = \pi(x+1)$, so

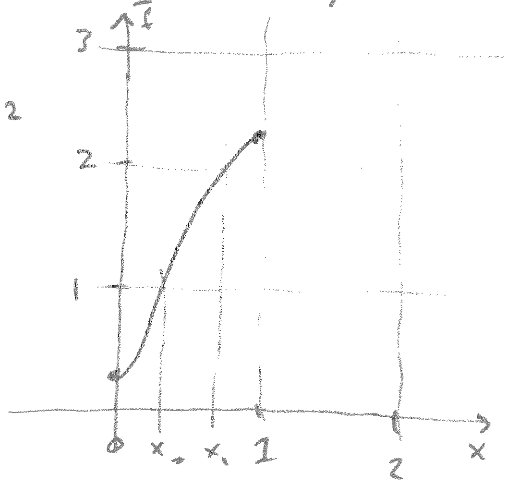
$f(\pi(x)) = f(\pi(x+1))$

$\Rightarrow$ $\pi(\bar{f}(x)) = \pi(\bar{f}(x+1))$ $\big(\pi(\bar{f}(x)) = f(\pi(x)) \big)$

$\Rightarrow$ $\bar{f}(x) = \bar{f}(x+1) + k(x)$ $k: \mathbb{R} \to \mathbb{Z}$.

But $\bar{f}(x)$ is continuous, so $k(x) = k$, constant

If $k \geq 2$



$\bar{f}(x_0) = 1$

$\bar{f}(x_1) = 2$

$\pi(x_0) \neq \pi(x_1)$

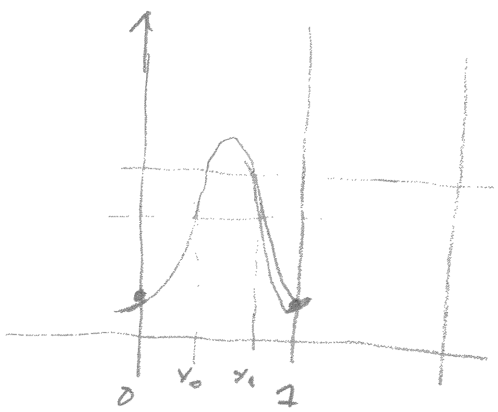
But $\pi(\bar{f}(x_0)) = \pi(\bar{f}(x_1)) = f(\pi(x_1))$

$\quad \shortparallel$

$f(\pi(x_0))$

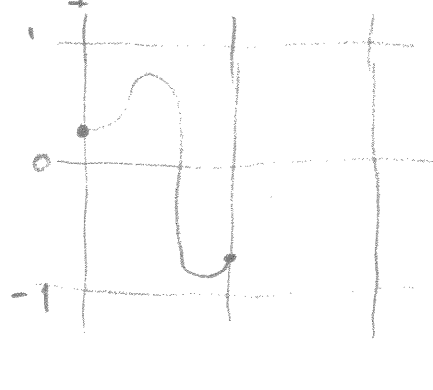
so then f is not a homeomorphism.

If $k = 0$, $\bar{f}(0) = \bar{f}(1)$



$\Rightarrow$ f is not injective.

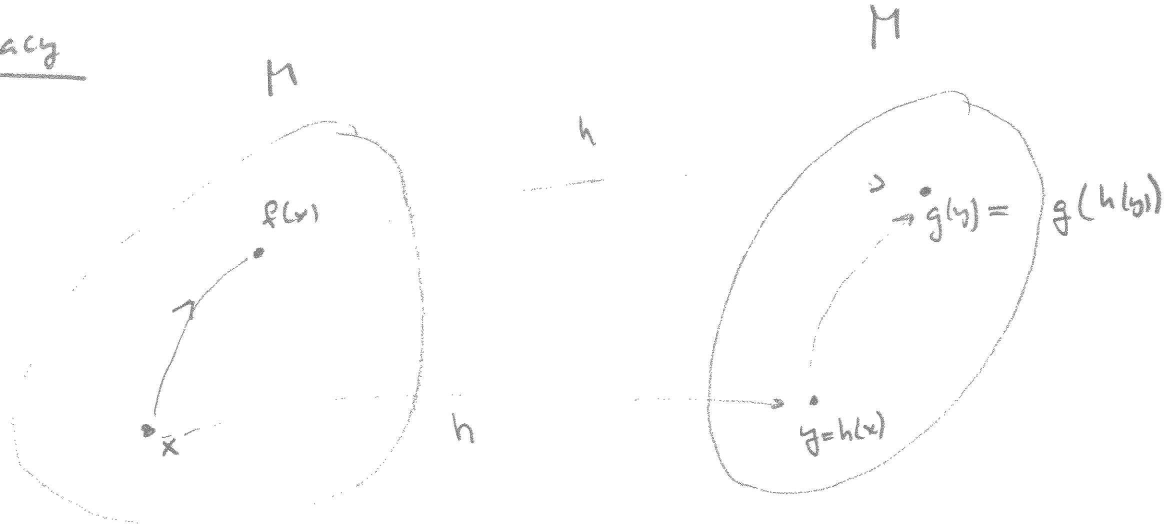
If $k \leq 0$



$\Rightarrow$ $f'(0) < 0$ at some point

$\Rightarrow$ not orientation preserving.

Conjugacy



Def Two diffeomorphisms $f, g : M \to M$ are topologically conjugate if there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$, i.e. $h(f(x)) = g(h(x))$ for all $x \in M$

and because h is a homeomorphism,

$$f(x) = h^{-1}(g(h(x)))$$

For flows $\Phi(t,x)$ we have

Def Φ_t and Ψ_t are topologically conjugate if there is $h : M \to M$ such that for all t

$$h(\Phi_t(x)) = \Psi_t(h(x))$$

weaker

Def The flows Φ_t and Ψ_t are topologically equivalent if there is a homeomorphism $h : M \to M$ such that h maps orbits of Φ_t onto orbits of Ψ_t, preserving orientation.

Ex consider

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - y^2 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x} = y \\ \dot{y} = -x - y^3 \end{cases}.$$

Both right hand sides have Jacobian

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which has characteristic equation

$\lambda^2 + 1 = 0$ and eigenvalues $\lambda = \pm i$.

This is not hyperbolic. What does the nonlinear perturbation do?

Check polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

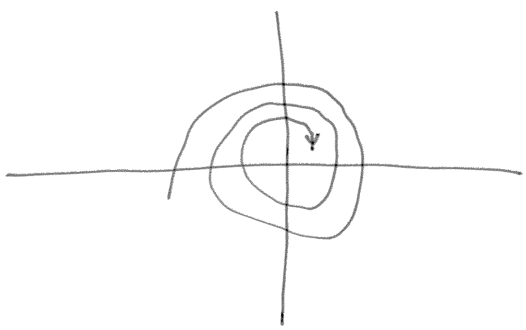
$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$
$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$\Rightarrow \begin{cases} \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = r\sin\theta \\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = -r\cos\theta - r^k\sin^k\theta \end{cases} \qquad k = 2, 3$$

$$\Rightarrow \begin{cases} \dot{r} = -r^k\sin^{k+1}\theta \\ r\dot{\theta} = -r - r^k\sin^k\theta\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \dot{r} = -r^k\sin^{k+1}\theta \\ \dot{\theta} = -1 - r^k\sin^k\theta\cos\theta. \end{cases}$$

For $k = 3$ we see that if $r < \frac{1}{2}$, $\dot{\theta} \in \left]-1-\frac{1}{8}, -1+\frac{1}{8}\right[$
and $\dot{r} = -r^3\sin^4\theta \leq 0$
and $\dot{r} = 0$ only if $\theta = n\pi$. Because $\dot{\theta} < -1 - \frac{1}{8}$,
θ will not stay at $\theta = n\pi$, and so $r(t)$ is strictly decreasing



For $k=2$ we cannot draw the same conclusion.
In fact we are told that orbits close to $x=y=0$ are closed. How can we see that?

Trick: try to find $P(x,y)$ with a local minimum at $x=y=0$ such that $\frac{d}{dt}P(x,y)=0$.

$$\frac{d}{dt}P(x,y) = \frac{\partial P}{\partial x}\dot{x} + \frac{\partial P}{\partial y}\dot{y} = 0 \quad \ldots$$

we want $\frac{\partial P}{\partial x} \sim -\dot{y}$, $\frac{\partial P}{\partial y} \sim \dot{x}$

Guess: try
$$\frac{\partial P}{\partial y} = g(x)\,y$$
$$\frac{\partial P}{\partial x} = g(x)\,(x+y^2)$$
(it did not work with $g(x)=1$)

$$\frac{\partial P}{\partial y} = g(x)\,y \Rightarrow P(x,y) = \frac{1}{2}g(x)y^2 + \phi(x)$$
$$\Rightarrow \frac{\partial P}{\partial x} = \frac{1}{2}g'(x)y^2 + \phi'(x) = g(x)x + g(x)y^2$$

Then
$$\begin{cases} \frac{1}{2}g'(x) = g(x) \\ \phi'(x) = x\,g(x) \end{cases} \Rightarrow \begin{array}{l} g(x) = c_1 e^{2x} \\ \phi(x) = c_1 \int x e^{2x} = \frac{c_1}{2}e^{2x}\left(x-\frac{1}{2}\right) + c_2 \end{array}$$

$$\Rightarrow P(x,y) = \frac{c_1}{2}e^{2x}y^2 + \frac{c_1}{2}e^{2x}\left(x-\frac{1}{2}\right) + c_2 \qquad \text{Take } c_1=2,\ c_2=0$$

$$\Rightarrow P(x,y) = e^{2x}\left(y^2+x-\frac{1}{2}\right)$$

We have then $\frac{d}{dt}P(x(t),y(t)) = 0$.

Make a Taylor expansion around $x=y=0$.

$$P(x,y) = \left(1+2x+2x^2+O(x^3)\right)\left(y^2+x-\frac{1}{2}\right) =$$
$$= \left(1+2x+2x^2\right)\left(y^2+x-\frac{1}{2}\right) + O(x^2+y^2)$$
$$= -\frac{1}{2} - x - x^2 + x + 2x^2 + 2x^3 + y^2 + 2xy^2 + 2x^2y^2 + O(|x|^3+|y|^3)$$
$$= -\frac{1}{2} + x^2 + y^2 + O(|x|^3+|y|^3)$$

Ex The Lotka-Volterra system

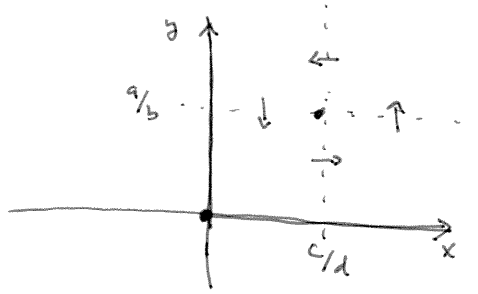
$$\begin{cases} \dot{x} = ax - bxy = x(a-by) & \text{prey} \\ \dot{y} = -cy + dxy = -y(c-dx) & \text{predator} \end{cases}$$

Equilibrium points: $x=y=0$ or $x=\frac{c}{d}$ $y=\frac{a}{b}$

At $x=y=0$:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + xy\begin{pmatrix} -b \\ d \end{pmatrix}$$

Note only interested in $x \geq 0$, $y \geq 0$



Stable manifold $\{(0,s),\ s \in \mathbb{R}^+\}$

unstable manifold $\{(s,0),\ s \in \mathbb{R}^+\}$

At $x=\frac{c}{d}$, $y=\frac{a}{b}$, let $x=\frac{c}{d}+\xi$, $y=\frac{a}{b}+\eta$

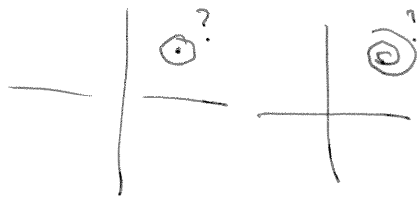
$$\Rightarrow \quad \dot{\xi} = b\left(\frac{c}{d}+\xi\right)\left(\frac{a}{b}-\frac{a}{b}-\eta\right) = -b\left(\frac{c}{d}+\xi\right)\eta = -\frac{bc}{d}\eta - b\xi\eta$$

$$\dot{\eta} = -d\left(\frac{a}{b}+\eta\right)\left(\frac{c}{d}-\frac{c}{d}-\xi\right) = +d\left(\frac{a}{b}+\eta\right)\xi = \frac{ad}{b}\xi + d\xi\eta$$

Here the Jacobian is $\begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{pmatrix}$ with char eq:

$$\lambda^2 + \frac{bc}{d}\frac{ad}{b} = 0 \quad \Rightarrow \quad \lambda = \pm i\sqrt{ac}$$

Not hyperbolic.



If $x=x(t)$, $y=y(t)$, we find

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y(c-dx)}{x(a-by)} \quad \Rightarrow \quad \frac{a-by}{y}dy = \frac{c-dx}{x}dx$$

$$\left(\frac{a}{y}-b\right)dy = \left(\frac{c}{x}-d\right)dx$$

$$a\log y - by = c\log x - dx + \underbrace{\log K}_{C}$$

$$\Rightarrow \qquad \underline{y^a e^{-y} = K x^{-c} e^{-x}}$$

$$a \log y - by + c \log x - d x = \log K$$

$$y = \frac{a}{b} + \eta \qquad x = \frac{c}{d} + \zeta$$

$$a \lg\left(\frac{a}{b} + \eta\right) - b\left(\frac{a}{b} + \eta\right) + c \lg\left(\frac{c}{d} + \zeta\right) - d\left(\frac{c}{d} + \zeta\right) = \lg K$$

$$a \lg\left(\frac{a}{b}\right) + a \lg\left(1 + \frac{b}{a}\eta\right) - a - b\eta$$
$$+ c \log\frac{c}{d} + c \lg\left(1 + \frac{d}{c}\zeta\right) - c - d\zeta = \lg K$$

$$a \lg\left(\frac{a}{b}\right) + a\left(\frac{b}{a}\eta + \left(\frac{b}{a}\eta\right)^2 + \dots\right) - a - b\eta$$
$$+ c \lg\frac{c}{d} + c\left(\frac{d}{c}\zeta + \left(\frac{d}{c}\zeta\right)^2 + \dots\right) - c - d\zeta = \lg K$$

$$\underline{\frac{b^2}{a}\eta^2 + \frac{d^2}{c}\zeta^2} = \lg K - a \lg\left(\frac{a}{b}\right) + a - c \lg\frac{c}{d} + c \quad + O(\zeta^3, \eta^3)$$

ellipse.