Solutions to some exercises

These notes contain answers/solutions to some of the given problems. In particular there will be comments and alternative solutions to problems that were presented in the lectures.

- **T2.10:** The purpose of this was to show a version of Gronwall's inequality. Lemma 2.7 states that if

$$\psi(t) \quad \leq \quad \alpha(t) + \int_0^t \beta(s)\psi(s)\,ds \tag{1}$$

with $\beta(t) \geq 0$, then

$$\psi(t) \quad \leq \quad \alpha(t) + \int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds\,. \tag{2}$$

The statement holds for any interval $t \in [0,T]$ where the inequality (1) is holds. The proof of this is given in the text. But the exercise is to prove that if α is a non-decreasing function, *i.e.* if $\alpha(s) \leq \alpha(t)$ whenever $s < t$, then

$$\psi(t) \leq \alpha(t)e^{\int_0^t \beta(s)\,ds} \tag{3}$$

The proof that was presented to us by Manuel was based on the idea of relating this problem to one that I had presented earlier, where $\alpha > 0$ was a constant. It worked fine, but required some rather clever tricks to do. Here is an alternative proof that also works fine.

We start by the inequality (2). Because α is non-decreasing, and both β and the exponential are positive, the integral can be estimated as

$$\begin{aligned}\int_0^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds \quad &\leq \quad \int_0^t \alpha(t)\beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds \\ &= \quad \alpha(t)\int_0^t \beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds \qquad (4)\end{aligned}$$

But

$$\beta(s)\exp\left(\int_s^t \beta(r)\,dr\right) \quad = \quad -\frac{d}{ds}\exp\left(\int_s^t \beta(r)\,dr\right) \tag{5}$$

and therefore

$$\int_0^t \beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds \quad = \quad e^{\int_0^t \beta(s)\,ds} - 1 \tag{6}$$

Now we plug this into the inequality (2) and find

$$\begin{aligned}\psi(t) \quad &\leq \quad \alpha(t) + \alpha(t)\left(e^{\int_0^t \beta(s)\,ds} - 1\right) \\ &= \quad \alpha(t)e^{\int_0^t \beta(s)\,ds} \qquad (7)\end{aligned}$$

which is the announced result.