2. (i) $\lim\limits_{x \to \infty} \dfrac{\ln(x^3+2x^2)}{5\ln(4x^5)} = \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x \to \infty} \frac{\frac{1}{x^3+2x^2} \cdot (3x^2+4x)}{\frac{5}{\cancel{4x^5}} \cdot \cancel{20}^{5}\cancel{x^4}} =$$

$$= \lim_{x \to \infty} \frac{3x^2+4x}{(x^3+2x^2) \cdot \frac{25}{x}} =$$

$$= \lim_{x \to \infty} \frac{\cancel{x^3}\left(3+\frac{4}{x}\right)}{25\cancel{x^3}\left(1+\frac{2}{x}\right)} = \underline{\underline{\frac{3}{25}}}$$

(ii) $\dfrac{\sqrt{1+\sin(x)}-\sqrt{1-\sin(x)}}{\tan(x)} =$

$$= \frac{\left(\sqrt{1+\sin(x)}-\sqrt{1-\sin(x)}\right)\left(\sqrt{1+\sin(x)}+\sqrt{1-\sin(x)}\right)}{\tan(x)\left(\sqrt{1+\sin(x)}+\sqrt{1-\sin(x)}\right)} =$$

$$= \frac{\cancel{1}+\sin(x)-(\cancel{1}-\sin(x))}{\frac{\sin(x)\left(\sqrt{1+\sin(x)}+\sqrt{1-\sin(x)}\right)}{\cos(x)}} =$$

$$= \frac{2\cancel{\sin(x)}}{\frac{\cancel{\sin(x)}\left(\sqrt{1+\sin(x)}+\sqrt{1-\sin(x)}\right)}{\cos(x)}} \xrightarrow[x \to 0]{}$$

$$\xrightarrow[x \to 0]{} \frac{2}{\frac{\sqrt{1}+\sqrt{1}}{1}} = \underline{\underline{1}}$$

(iii) Låt $y = \left(\frac{x+1}{x-2}\right)^{2x-1}$. Då

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} (2x-1)\ln\left(\frac{x+1}{x-2}\right) \leftarrow [\infty\cdot 0] =$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-2}\right)}{\frac{1}{2x-1}} \leftarrow \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to\infty} \frac{\frac{\cancel{x-2}}{x+1}\cdot\left(\frac{1\cdot(x-2)-(x+1)\cdot 1}{(x-2)^{\cancel{2}}}\right)}{-\frac{2}{(2x-1)^2}} =$$

$$= \lim_{x\to\infty} \frac{\frac{+3}{(x+1)(x-2)}}{\frac{+2}{(2x-1)^2}} = \lim_{x\to\infty} \frac{3(2x-1)^2}{2(x+1)(x-2)} =$$

$$= \lim_{x\to\infty} \frac{3(2\cancel{x})^2\left(1-\frac{1}{2x}\right)^2}{2\cancel{x^2}\left(1+\frac{1}{x}\right)\left(1-\frac{2}{x}\right)} = \frac{3\cdot 2^2}{2} = \underline{\underline{6}}$$

$\therefore \lim_{x\to\infty} \ln(y) = 6 \Rightarrow \lim_{x\to\infty} y = e^6$ då e^x

är kontinuerlig i $x=6$.

Svar: $\underline{\underline{e^6}}$

3. $$\int \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}\,dx = \left\{\begin{array}{l} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{array}\right\} =$$

$$= \int \frac{\arctan(t)}{\cancel{t}(1+t^2)}\,2\cancel{t}\,dt = 2\int \frac{\arctan(t)}{1+t^2}\,dt$$

$$\int \frac{\arctan(t)}{1+t^2}\,dt = \left\{\begin{array}{l} \text{partiell} \\ \text{integration} \end{array}\right\} =$$

$$= (\arctan(t))^2 - \int \frac{\arctan(t)}{1+t^2}\,dt$$

$$\Rightarrow \quad 2\int \frac{\arctan(t)}{1+t^2}\,dt = (\arctan(t))^2 + C$$

$$\therefore \int \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}\,dx = (\arctan(\sqrt{x}))^2 + C$$

4. $(x^2+1)y' + x(y^2+y) = 0 \iff$

$$\iff \frac{1}{y(y+1)}\, y' = -\frac{x}{x^2+1} \Rightarrow$$

$$\Rightarrow \int \frac{dy}{y(y+1)} = -\int \frac{x}{x^2+1}\, dx = -\frac{1}{2}\ln|x^2+1| + C$$

Partialbråksuppdelning:

$$\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \iff 1 = A(y+1) + By$$

$\underline{y=0}$: $1 = A$

$\underline{y=-1}$: $1 = -B \iff B = -1$

$$\int \frac{dy}{y(y+1)} = \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy =$$

$$= \ln|y| - \ln|y+1| = \ln\left|\frac{y}{y+1}\right|$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln\left((x^2+1)^{-1/2}\right) + C$$

$$\Rightarrow \frac{y}{y+1} = \pm e^C \cdot (x^2+1)^{-1/2} = C\,(x^2+1)^{-1/2}$$

$$y = C(x^2+1)^{-1/2}(y+1) \iff$$

$$\iff y\left(1 - C(x^2+1)^{-1/2}\right) = C(x^2+1)^{-1/2}$$

$$\therefore \quad y(x) = \frac{C(x^2+1)^{-1/2}}{1 - C(x^2+1)^{-1/2}} = \frac{1}{C\sqrt{x^2+1} - 1}$$

$C = 0 \Rightarrow y = -1$ ok!

Lösningen existerar för alla x om $C < 0$

och för alla x utom x s.a. $\sqrt{x^2+1} \neq \frac{1}{C}$

om $C > 0$.

6. (i) <u>Homogenlösn.</u>: Kar. ekv.: $r^2 - 4r + 4 = 0$

$$\Rightarrow r = 2 \pm \sqrt{2^2 - 4} = 2$$

$$\Rightarrow \quad y_h(x) = (C_1 x + C_2)e^{2x}$$

(ii) <u>Partikulärlösn.</u>: Studera hjälpekvationen

$$u'' - 4u' + 4u = e^{2ix}$$

Låt ~~z~~ $u = ze^{2ix} \Rightarrow$

$$\Rightarrow u' = z'e^{2ix} + 2ize^{2ix} = e^{2ix}(z' + 2iz)$$

$$u'' = z''e^{2ix} + 4iz'e^{2ix} - 4ze^{2ix} =$$

$$= e^{2ix}(z'' + 4iz' - 4z)$$

$$u'' - 4u' + 4u = e^{2ix}(z'' + 4iz' - 4z) -$$

$$- 4e^{2ix}(z' + 2iz) + 4ze^{2ix} =$$

$$= e^{2ix}(z'' + 4iz' - \cancel{4z} - 4z' - 8iz + \cancel{4z}) =$$

$$= e^{2ix}(z'' + 4(-1+i)z' - 8iz) = e^{2ix}$$

$$\Rightarrow z'' + 4(-1+i)z' - 8iz = 1$$

$$\Rightarrow z_p = -\frac{1}{8i} = \frac{i}{8} \quad \Rightarrow$$

$$\Rightarrow \quad u_p = z_p e^{2ix} = \frac{i}{8}\left(\cos(2x) + i\sin(2x)\right) =$$

$$= -\frac{1}{8}\sin(2x) + i\cdot\frac{1}{8}\cos(2x)$$

$$y_p(x) = \operatorname{Re}(u_p) = -\frac{1}{8}\sin(2x)$$

$$\therefore \quad y(x) = y_h(x) + y_p(x) =$$

$$= (C_1 x + C_2)e^{2x} - \frac{1}{8}\sin(2x)$$

7. $\ln(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3)$

$\cos(x) = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$

$\sin(x) = x - \frac{x^3}{6} + \mathcal{O}(x^5)$

$$\frac{a(\ln(1+x))^3}{(b-\cos(x))\sin(x)} = \frac{a\left(x - \frac{x^2}{2} + \mathcal{O}(x^3)\right)^3}{\left(b - 1 + \frac{x^2}{2} + \mathcal{O}(x^4)\right)\left(x - \frac{x^3}{6} + \mathcal{O}(x^5)\right)}$$

$$= \frac{a\left(x^3 + \mathcal{O}(x^4)\right)}{(b-1)x - \frac{(b-1)}{6}x^3 + \frac{1}{2}x^3 + \mathcal{O}(x^5)}$$

Ser att för att ett gränsvärde skall existera då $x \to 0$ måste $b = 1$

$$\Rightarrow \frac{a(x^3 + \mathcal{O}(x^4))}{\frac{1}{2}x^3 + \mathcal{O}(x^5)} = \frac{a \cancel{x^3}(1 + \mathcal{O}(x))}{\cancel{x^3}\left(\frac{1}{2} + \mathcal{O}(x^2)\right)} \xrightarrow[x \to 0]{}$$

$$\xrightarrow[x \to 0]{} \frac{a}{\frac{1}{2}} = 2a = 1 \text{ om } a = \frac{1}{2}$$

$\therefore\ a = \frac{1}{2}$ och $b = 1$

MMG 200 Envariabelanalys

Tenta januari 2017 – Lösningar.

1) Bevis finns i boken.

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5) $f(x) = e^{\sin(x^2)}$ på $[-1, 1]$.

Beräkna $f'(x) = 2x\cos(x^2)\, e^{\sin(x^2)}$ (Kedjeregel).

$f'(x) = 0 \iff 2x\cos(x^2) = 0$ darför att $e^{\cdots} \neq 0$ om den här är i $\mathbb{R}$.

$2x\cos(x^2) = 0$ om $x = 0$ eller $\cos(x^2) = 0$.

$\cos(x^2) = 0 \iff x^2 = \frac{n+1}{2}\pi$ för något $n \in \mathbb{N} \cup \{0\}$

$\implies \left|\frac{(n+1)}{2}\pi\right| \geq \frac{\pi}{2}$. $\sqrt{\pi/2} > 1$.

Altså $f'(x) \neq 0 \quad \forall\, x \in [-1, 1] \setminus \{0\}$.

$f(\pm 1) = e^{\sin(1)} > 1$ darför att $0 < 1 < \pi/2$.

$\therefore$ Maximum av f antagit med $x = \pm 1$,
minimum av f antagit med $x = 0$, $f(0) = 1$.

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8. Fund. Sats. Analys: funktionen 1 är Kontinuerlig $\Rightarrow$
$f(x) = \int_0^x f'(t)\,dt \quad \forall\, x \in [0, 1]$. $\therefore f(x) = \int_0^x 1\,dt = x$.

Om en funktion g har samma 3 egenskaper,
gäller:
1) $f(x) - g(x) = h(x)$ är kontinuerlig i $[0,1]$
2) $h(x)$ är deriverbar i $]0,1[$.
3) $h'(x) = 0 \quad \forall\, x \in]0,1[\underset{\text{sats}}{\implies} h$ är <u>Konstant</u>.
4) $h(0) = 0$ tillsammens med 1) $\Rightarrow h(x) = 0 \ \forall\, x \in [0,1]$
$\Rightarrow f(x) = g(x) = x$.