Lösningar MMG200 envariabel analys 2017 augusti

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① Finns i boken!

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② (i) $\lim_{x\to\pi} \frac{\cos(2x)}{x^2-\pi^2}$ finns inte! $\cos(2x) \to 1$

$x^2-\pi^2 \to 0.$

(ii) $$\lim_{x\to\infty} \frac{\ln(20x^2+3x+5)}{\ln(x^{10})} = \lim_{x\to\infty} \frac{\ln\left(x^2\left(20+3/x+5/x^2\right)\right)}{10\ln(x)}$$

$$= \lim_{x\to\infty} \frac{2\ln(x) + \ln(20+3/x+5/x^2)}{10\ln(x)} = \frac{1}{5}.$$

(iii) $$\lim_{x\to 0} \frac{\ln(1+x^2) - \sin^2(x)}{1-\cos(x^2)} = \lim_{x\to 0} \frac{\sum_{k\geq 1} \frac{(x^2)^k}{k} - \left(\sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)^2}{1 - \sum_{k\geq 0} \frac{(-1)^k (x^2)^{2k}}{(2k)!}}$$

↑ använd Taylor-utveckling

$$= \lim_{x\to 0} \frac{x^2 + \frac{x^4}{2} + \cdots - \left(x - \frac{x^3}{3!} + \cdots\right)^2}{1 - \left(1 - \frac{x^4}{2!} + \cdots\right)} =$$

$$\lim_{x\to 0} \frac{\frac{x^4}{2} - \left(\frac{-2x^4}{3!} + \cdots\right)}{\frac{x^4}{2!} + \cdots} = \lim_{x\to 0} \frac{\frac{5x^4}{6} + \cdots}{\frac{x^4}{2} + \cdots} = \frac{5}{12}$$

Då allt ... betyder att det försvinner när $x\to 0$.

3. $\int f(x)\,dx = \int \frac{1}{\sqrt{x}\,(1+\sqrt[3]{x})}\,dx = \left\{ \begin{array}{l} x = u^6 \\ dx = 6u^5\,du \end{array} \right\} =$

$$= \int \frac{1}{(u^6)^{1/2}\left(1+(u^6)^{1/3}\right)}\, 6u^5\,du =$$

$$= 6\int \frac{u^2}{1+u^2}\,du = 6\int \left(1 - \frac{1}{1+u^2}\right) du =$$

$$= 6\left(u - \arctan(u)\right) = \left\{ u = x^{1/6} \right\} =$$

$$= 6x^{1/6} - 6\arctan\left(x^{1/6}\right) + C$$

(4) $f(x) = \ln(\cos(\pi x) + 2)$.

$f'(x) = \dfrac{-\pi \sin(\pi x)}{\cos(\pi x) + 2}$. Försvinner när

$\sin(\pi x) = 0 \iff x \in \mathbb{Z}$.

Låt $x = k \in \mathbb{Z}$.

$$f(x) = \ln(\cos(\pi x) + 2)$$
$$f(k) = \ln(\cos(\pi k) + 2)$$
$$= \ln((-1)^k + 2)$$
$$= \begin{cases} \ln(3) & k \text{ jämn} \\ \ln(1) = 0 & k \text{ udda} \end{cases}$$

$\therefore$ Extrempunkter är alla punkter i $\mathbb{Z}$.

Minima om $k \in \mathbb{Z}$ är udda, då är funktionen $= 0$, maxima om $k \in \mathbb{Z}$ är jämn, då är funktionen $= \ln(3)$.

5. Låt $f(x) = \frac{1}{\sqrt{x^2+x}}$.

$$\Rightarrow V = \int_1^2 \pi f(x)^2 \, dx = \pi \int_1^2 \frac{1}{x^2+x} \, dx =$$

$$= \pi \int_1^2 \frac{1}{x(x+1)} \, dx =$$

$$= \left\{ \begin{array}{l} \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Leftrightarrow 1 = A(x+1) + Bx \\ \Rightarrow A = 1, \; B = -1 \end{array} \right\} =$$

$$= \pi \int_1^2 \left(\frac{1}{x} - \frac{1}{1+x} \right) dx = \pi \Big[\ln|x| - \ln|1+x| \Big]_1^2 =$$

$$= \pi \left[\ln\left| \frac{x}{1+x} \right| \right]_1^2 = \pi \left(\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right) \right) =$$

$$= \pi \ln\left(\frac{4}{3}\right) \quad \text{v.e.}$$

6. (i) <u>Homogen lösn.</u>: Kar. ekv.: $r^2 - 2r + 1 = 0$

$$\Rightarrow \quad r = 1 \pm \sqrt{1-1} = 1$$

$$\therefore \quad y_h(x) = (C_1 + C_2 x)e^x$$

(ii) <u>Partikulärlösn.</u>: $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

$$\Rightarrow \quad y'' - 2y' + y = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$y_{p_1} = \frac{1}{2}$$

Betrakta: $u'' - 2u' + u = -\frac{1}{2}e^{2ix}$

Låt $u = ze^{2ix} \Rightarrow u' = z'e^{2ix} + 2ize^{2ix}$

$$\Rightarrow u'' = z''e^{2ix} + 4iz'e^{2ix} - 4ze^{2ix}$$

$$\Rightarrow u'' - 2u' + u = (z'' + 4iz' - 4z - 2z' - 4iz + z)e^{2ix}$$

$$= \left(z'' + (-2+4i)z' - (3+4i)z\right)\cancel{e^{2ix}} = -\frac{1}{2}\cancel{e^{2ix}}$$

$$\Rightarrow z_p = -\frac{1}{2}\cdot\frac{1}{(-3-4i)} = \frac{1}{6+8i}\cdot\frac{6-8i}{6-8i} = \frac{6}{100} - \frac{8i}{100}$$

$$\Rightarrow u_p = z_p e^{2ix} = \left(\frac{6}{100} - \frac{8i}{100}\right)\left(\cos(2x) + i\sin(2x)\right) =$$

$$= \frac{6}{100}\cos(2x) + \frac{8}{100}\sin(2x) + i\cdot(\ldots)$$

$$y_{p_2} = \operatorname{Re}(u_p) = \frac{1}{100}\left(6\cos(2x) + 8\sin(2x)\right)$$

$$\Rightarrow \quad y_p = y_{p_1} + y_{p_2} = \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

$$\therefore \quad y = y_h + y_p = (C_1 + C_2 x)e^x + \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

(iii) <u>Begynnelsevillkor</u>:

$$y(0) = C_1 + \frac{1}{2} + \frac{3}{50} = 1 \Leftrightarrow C_1 = 1 - \frac{28}{50} = \frac{22}{50} = \frac{11}{25}$$

$$y'(x) = \frac{11}{25}e^x + C_2 e^x + C_2 x e^x - \frac{6}{50}\sin(2x) + \frac{8}{50}\cos(2x)$$

$$y'(0) = \frac{11}{25} + C_2 + \frac{8}{50} = \frac{3}{5} + C_2 = 2 \Leftrightarrow C_2 = \frac{7}{5}$$

$$\therefore \quad y = \left(\frac{11}{25} + \frac{7}{5}x\right)e^x + \frac{1}{2} + \frac{3}{50}\cos(2x) + \frac{4}{50}\sin(2x)$$

(7) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

nästa steg ↓↓↓

Altså: $$\lim_{h \to 0} \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h} \left\{ \frac{\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)}{\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)} \right.$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+x+h} - \frac{1}{1+x}}{h\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)} = \lim_{h \to 0} \frac{\frac{1+x-(1+x+h)}{(1+x+h)(1+x)}}{h\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h(1+x+h)(1+x)\left(\frac{1}{\sqrt{1+x+h}} + \frac{1}{\sqrt{1+x}}\right)}$$

$$= \frac{-1}{(1+x)^2 \left(\frac{2}{\sqrt{1+x}}\right)} = \frac{-1}{2(1+x)^{3/2}}.$$

8. $$\frac{1}{e^x-1} - \frac{a\ln(1+2x)}{x\tan(x)} =$$

$$= \frac{x\tan(x) - a(e^x-1)\ln(1+2x)}{x(e^x-1)\tan(x)} =$$

$$= \frac{x\left(x+\mathcal{O}(x^3)\right) - a\left(x+\frac{1}{2}x^2+\mathcal{O}(x^3)\right)\left(2x-\frac{1}{2}4x^2+\mathcal{O}(x^3)\right)}{x\left(x+\mathcal{O}(x^2)\right)\left(x+\mathcal{O}(x^3)\right)}$$

$$= \frac{x^2+\mathcal{O}(x^4) - a\left(2x^2-2x^3+x^3+\mathcal{O}(x^4)\right)}{x^3+\mathcal{O}(x^4)} =$$

$$= \frac{(1-2a)x^2 + ax^3 + \mathcal{O}(x^4)}{x^3+\mathcal{O}(x^4)} \Rightarrow$$

$\Rightarrow$ För att ett gränsvärde då $x \to 0$ ska kunna existera måste $\underline{\underline{a = 1/2}}$

I så fall:

$$\frac{1}{e^x-1} - \frac{1}{2}\frac{\ln(1+2x)}{x\tan(x)} = \frac{\frac{1}{2}x^3+\mathcal{O}(x^4)}{x^3+\mathcal{O}(x^4)} =$$

$$= \frac{\frac{1}{2}+\mathcal{O}(x)}{1+\mathcal{O}(x)} \to \underline{\underline{\frac{1}{2}}} \quad \text{då } x \to 0$$