MATEMATIK
Göteborgs universitet

Dugga 2

Envariabelanalys, hösten 2017

Fel svar = -1 P. Rätt svar = + 1P. Total Poäng blir maximum av noll och poäng som ni har fått på duggan. Altså maximum är 10P total, minimum är 0P total.

Skriv ditt namn och personnummer:

(2P möjligt) Bestäm om gränsvärdet finns och i så fall beräkna gränsvärdet:

$$\lim_{x \to 0} x \tan\left(\frac{1}{x}\right)$$

$$\lim_{x \to \frac{\pi}{2}} \frac{e^x - \sin(x)}{x - \frac{\pi}{2}}$$

Oh, the first one is really tricky. The limit does not exist, because every time $x = \frac{1}{(k\pi + \pi/2)}$ for an integer k, the tangent is not defined. This is because cosine vanishes there. Hence, there are always points closer and closer to 0 at which

$$x \tan(1/x)$$

is not defined. So, the definition of limit can never be fulfilled.

Now, the second one turns out also to be not defined. This is because the numerator tends to

$$e^{\pi/2} - 1$$

whatever that is (it is not zero because $e^{\pi/2} > e^1$ since $\pi/2 > 1$ and $e^1 > 2 > 1$.) However, the denominator does tend to zero. So that one doesn't exist either. This is why you should not guess. The only way to succeed is to know your definitions and theorems!

(3P möjligt) Bestäm om följande talfjölder har ett gränsvärde och i så fall bestäm gränsvärdet

$$x_0 = 2^0, \quad x_1 = 2^0 + 2^{-1}, \quad x_2 = 2^0 + 2^{-1} + 2^{-2}, \quad x_n = x_{n-1} + 2^{-n}.$$

$$x_1 = 1, \quad x_2 = \left(1 + \frac{1}{2}\right)^2, \quad x_3 = \left(1 + \frac{1}{3}\right)^3, \quad x_n = \left(1 + \frac{1}{n}\right)^n.$$

$$x_1 = 1, \quad x_n = 1 + x_{n-1}.$$

The first one is an example of a geometric progression, and you might have forgotten, but I showed you how to handle these. It was a neat trick:

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

If you forget how this works, just multiply

$$(1 + x + x^2 + \ldots + x^n)(1 - x)$$

and show you get $1 - x^{n+1}$. So, we see that

$$\lim_{n\to\infty} 1 + (1/2) + (1/2)^2 + \ldots + (1/2)^n = \lim_{n\to\infty} \frac{1 - (1/2)^{n+1}}{1 - 1/2} = 2.$$

This is because

$$\lim_{n\to\infty} (1/2)^{n+1} = 0.$$

Now we look at the next guy. Well, I certainly hope you remember

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Finally, the third sequence looks like

$$1, 2, 3, 4, 5, \ldots$$

It doesn't converge because it is just getting bigger and bigger.

(3 P möjligt) Bestäm (1) största mängden i $\mathbb{R}$ där funktionen är deriverbar och (2) beräkna derivatan i den mängden:

$$f(x) = \sqrt{e^{\cos(x)}}$$
$$f(x) = \ln(2^x)$$
$$f(x) = \sin(\tan(x))$$

For the first one, we note that e^{whatever} is always positive. So, we can always square root it, no problems there. Hence the first function is differentiable everywhere. All of $\mathbb{R}$. We take its derivative using the chain rule (Yo Dawg):

$$\left(\sqrt{e^{\cos(x)}}\right)' = \frac{1}{2}\frac{-\sin(x)e^{\cos(x)}}{\sqrt{e^{\cos(x)}}}.$$

Next function. Well, $2^x > 0$ for all $x \in \mathbb{R}$. So there's no problems with taking ln of it. Hence second function is differentiable in all of $\mathbb{R}$ and its derivative is

$$(\ln(2^x))' = \frac{1}{2^x}\ln(2)2^x = \ln(2).$$

You could get at this in an easier way by remembering the rules for logs:

$$\ln(2^x) = x\ln(2).$$

Then you see more easily that the function is both totally fine everywhere as well as it's got derivative $\ln(2)$.

Third function is fine except at the points where tangent has its little "problems." One could think of it like tangent has a temper tantrum whenever $x = k\pi + \frac{\pi}{2}$. It blows up there! So, on $\mathbb{R}$ without all of these points, the function is totes adorbs, and has derivative

$$(\sin(\tan(x))' = \cos(\tan(x))\sec^2(x).$$

(2P möjligt) Hitta alla globala max och min punkter till funktionerna (eller bestäm om funktionen saknar max eller min):

$$f(x) = \sqrt{e^{\cos(x)}}$$

$$f(x) = \sin(\tan(x))$$

Well, what really matters with the first one is what's going on with the cosine. When it's at its max, the whole thing will be at its max, and similarly when its at its min, the whole thing will be at its min. So, we see that the max points for

$$\sqrt{e^{\cos(x)}}$$

are the points $2k\pi$ for all $k \in \mathbb{Z}$. At those points the function has value $\sqrt{e}$. The min points are for the points of the form (odd integer)π, which can be written like $(2k+1)\pi$ for all $k \in \mathbb{Z}$, at which the function has value

$$\sqrt{1/e} = \frac{1}{\sqrt{e}}.$$

As for the other guy, well... it may look like it's missing max and min points at first, but it's not. Sine is at its largest when the stuff inside is equal to an even multiple of π plus $\frac{\pi}{2}$. So, at all points of the form

$$x = \arctan(2k\pi + \pi/2),$$

the function $\sin(\tan(x))$ assumes its maximum value which is 1. Sine is at its smallest when the stuff inside is equal to an odd multiple of π plus $\frac{\pi}{2}$. So, whenever

$$x = \arctan((2k+1)\pi + \pi/2),$$

then $\sin(\tan(x))$ has its minimum value. That is, all points x whose tangent is $(2k+1)\pi + \pi/2$ give us minimum.