MATEMATIK
Göteborgs Universitet
Tentamen

Hjälpmedel: Inga, inte ens miniräknare
Datum: 27 augusti 2018 kl. 08.30–12.30
Telefonvakt: Oskar Allerbo

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MMG200 Envariabelsanalys

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
Betygsgränser: G: 14-21 poäng, VG: 22-25 poäng (22-31 poäng inklusive eventuella duggapoäng).
Lösningar läggs ut på kursens webbsida första vardagen efter tentamensdagen. Resultat meddelas via Ladok ca. tre veckor efter tentamenstillfället.

Varje uppgift omfattar 3 poäng utom uppgift 4 som omfattar 4 poäng. Till samtliga uppgifter skall fullständiga lösningar inlämnas. **Endast svar ger inga poäng**. Motivera och förklara så väl du kan.

Del 1

1. Bestäm om gränsvärdet finns och i så fall beräkna gränsvärdet:

(a)

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right) \quad (1p)$$

We consider what is happening with each term. $x \to \infty$ is clear enough. When $x \to \infty$, then $\frac{1}{x} \to 0$. The tangent of 0 is zero. So, it's a bit fuzzy what will happen. However, we can remember that

$$\tan(1/x) = \frac{\sin(1/x)}{\cos(1/x)}.$$

Then our limit looks like

$$\lim_{x\to\infty} \frac{x\sin(1/x)}{\cos(1/x)}.$$

This resembles a "standard limit." To turn it into that standard limit, let us change variables, defining $t = 1/x$. Then when $x \to \infty$, $t \to 0$. So our limit is

$$\lim_{t\to 0} \frac{1}{t}\frac{\sin(t)}{\cos(t)} = \lim_{t\to 0} \frac{\sin(t)}{t\cos(t)}.$$

Since

$$\lim_{t\to 0} \frac{\sin(t)}{t} = 1 = \lim_{t\to 0} \cos(t)$$

the limit exists and is equal to one.

(b)

$$\lim_{x\to\infty} \sqrt{x^2+3x-9} - \sqrt{x^2-3x+9} \quad (1p)$$

We do the completing the square trick for this one!

$$\sqrt{x^2+3x-9} - \sqrt{x^2-3x+9} = \frac{(\sqrt{x^2+3x-9} - \sqrt{x^2-3x+9})(\sqrt{x^2+3x-9} + \sqrt{x^2-3x+9})}{\sqrt{x^2+3x-9} + \sqrt{x^2-3x+9}}$$

$$= \frac{x^2+3x-9-(x^2-3x+9)}{\sqrt{x^2+3x-9} + \sqrt{x^2-3x+9}} = \frac{6x-18}{\sqrt{x^2+3x-9} + \sqrt{x^2-3x+9}}.$$

The leading term in the numerator is $6x$. The denominator is growing like $2x$. So the limit exists and is equal to three.

(c)

$$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) \quad (1p)$$

For this one, we use the old sandwich theorem. $x \to 0$, whereas cosine oscillates around. OBS! It stays between ± 1. Since

$$|\cos(1/x)| \le 1 \quad \forall x,$$

this shows that

$$0 \le \lim_{x\to 0} |x\cos(1/x)| \le \lim_{x\to 0} |x| = 0.$$

It therefore follows that the limit in question exists and is equal to zero.

2. (a) Bestäm om gränsvärdet finns och i så fall beräkna gränsvärdet: (1p)

$$\lim_{x\to 0} x\cot(x)$$

Here we just need to remember what the cotangent is. It is $\cos(x)/\sin(x)$. So, the limit in question is

$$\lim_{x\to 0} \frac{x\cos(x)}{\sin(x)}.$$

This is very similar to (1.a). (You're welcome!) Since

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \implies \lim_{x\to 0} \frac{x}{\sin(x)} = 1,$$

and

$$\lim_{x\to 0} \cos(x) = 1$$

the limit in question exists and is equal to one.

(b) Hitta alla lösningar i $\mathbb{C}$ till ekvationen: $x^4 + 2x^2 + 1 = 0$. (2p)
This is a quadratic equation in the variable $t = x^2$,

$$t^2 + 2t + 1 = 0.$$

This equation factors really nicely as

$$(t+1)^2 = 0.$$

So, there is a "double root" at $t = -1$. Since $x^2 = t$, this means that the only solutions are

$$x = \pm i,$$

which in complex polar coordinates is the same as

$$x = e^{i\pi/2} \text{ or } x = e^{3\pi i/2}.$$

3. Låt

$$f(x) = \sin(e^{\cos(x)}).$$

(a) Beräkna $f'(x)$. (1p)
We put a function in your function so you can derive while you derive (the chain rule). Repeated use shows that the derivative

$$f'(x) = \cos(e^{\cos(x)})e^{\cos(x)}(-\sin(x)).$$

(b) Hitta funktionens maximum i $\mathbb{R}$. (2p)

It's probably easiest to just look at $f(x)$ directly and think about its constituent parts. The sine function oscillates between ± 1. The same is true for the cosine. The sine will have maximums when its argument is equal to $\frac{\pi}{2} + 2k\pi$, for $k \in \mathbb{Z}$. Now let's look at $e^{\cos(x)}$. Since $-1 \leq \cos(x) \leq 1$ for all x, it follows that $e^{-1} \leq e^{\cos(x)} \leq e^1$ for all x. Hopefully you remember that e is approximately 2.7. On the other hand, e^{-1} is bigger than 1/3 but smaller than 1/2. So, the only number of the form $\frac{\pi}{2} + 2k\pi$ which is between e^{-1} and e is $\frac{\pi}{2}$. (This is a number somewhere around 1.5, since $\pi = 3.14.......$).

So, the only maximum of our function occurs when $e^{\cos(x)} = \frac{\pi}{2}$. Solving for x we obtain

$$x = \arccos\left(\ln\left(\frac{\pi}{2}\right)\right).$$

4. (Obs! 4Poäng!)

Formulera och bevisa kedjeregeln. (4p)

This is a theory item, and its proof is contained in the textbook!

Del 2

5. (a) Beräkna $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x^2-1}dx$. (1 p.)

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x^2-1}dx = -\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1+x+1-x}{(1-x)(1+x)}dx = -\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{1-x} + \frac{1}{1+x})dx =$$

$$-\frac{1}{2}[-\ln(1-x) + \ln(1+x)]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{2}(-\ln(1/2) + \ln(3/2) + \ln(3/2) - \ln(1/2))$$

$$= -\frac{1}{2} \cdot 2\ln(3) = -\ln(3).$$

(b) Hitta dem primitiva funktionerna $\int \arctan(x)dx$. (2 p.)

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2}dx =$$

$$x\arctan(x) - \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

6. Ytan begränsad av kurvorna $y = x^2$ och $y = 3 - 2x$ roterar kring x-axeln. Beräkna rotationskroppens volym. (3 p.)

Kurvurna skär varandra när $x^2 = 3 - 2x$, i.e. när $x^2 + 2x - 3 = 0$. Röterna till denna ekvation är $x = -3$ och $x = 1$. Eftersom båda funktioner är icke-negativa på $[-3, 1]$ och den andra ligger över den första, blir kropens volum skilnad mellan den som man får genom att rotera undergrafen till $y = 3 - 2x$ och undergrafen till $y = x^2$.

Dvs.

$$\int_{-3}^{1} \pi(3-2x)^2dx - \int_{-3}^{1} \pi(x^2)^2dx = \pi\int_{-3}^{1}(9 - 12x + 4x^2)dx + [\frac{1}{5}\pi x^5]_{-3}^{1} =$$

$$\pi[9x - 6x^2 + \frac{4}{3}x^3]_{-3}^{1} - \frac{1}{5}\pi(1+3^5) = \pi(9\cdot(1+3) - 6(1-9) + \frac{4}{3}(3^3+1) - \frac{1}{5}(1+3^5))$$

$$= \pi(36 + 48 + \frac{4}{3}28 - \frac{1}{5}244) = \pi(84 + 28 + 9 + 1/3 - 48 - 4/5) = \pi 72\frac{8}{15} = \pi 1088/15.$$

7. (a) Lös differentialekvationen $xy' + y = 1$. (2 p.)

$(xy)' = 1$, så $xy = x + C$, i.e. $y = 1 + C/x$. (Man får dela när $x \neq 0$, och när x=0 är lösning ej definierat.)

(b) Lös begynnelsevärdesproblemet $xy' + y = 1$, $y(1) = 2$. (1 p.)

För allmän lösning se (a). Då $y(1) = 2$, har vi $2 = 1 + C$, i.e. $C = 1$. Lösningen blir $y = 1 + 1/x$.

8. (a) Ange Maclaurins serie till $f_1(x) = \frac{1}{1-x}$. (1 p.)

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$.

(b) Ange Maclaurins serie till $f_2(x) = \frac{x}{1-x}$ och $f_3 = \frac{1}{1+x}$. (1 p.)

$\frac{x}{1-x} = x\frac{1}{1-x} = x + x^2 + x^3 + \ldots = \sum_{k=1}^{\infty} x^k$.

$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k$.

(c) Ange Maclaurins serie till $f_4(x) = \ln(1 + x)$. (1 p.)

$\ln(1 + x) = \int_0^x \frac{1}{1+x} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}x^k$.

Lycka till!

Julie och Maria