MATEMATIK
Göteborgs Universitet
Tentamen

Hjälpmedel: Inga, inte ens miniräknare
Datum: 2018 kl. 08.30–12.30
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MMG200 Envariabelsanalys

Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Skriv del 1 och del 2 i separata omslag. Fyll i omslagen ordentligt.

Betygsgränser: G: 14-21 poäng, VG: 22-25 poäng (22-31 poäng inklusive eventuella duggapoäng).

Lösningar läggs ut på kursens webbsida första vardagen efter tentamensdagen. Resultat meddelas via Ladok ca. tre veckor efter tentamenstillfället.

Varje uppgift omfattar 3 poäng utom uppgift 4 som omfattar 4 poäng. Till samtliga uppgifter skall fullständiga lösningar inlämnas. **Endast svar ger inga poäng**. Motivera och förklara så väl du kan.

Del 1

1. Bestäm om gränsvärdet finns och i så fall beräkna gränsvärdet:

$$\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{x^2} \quad (1p)$$

Let's just look at each piece separately and think about what's happening. As $x \to \infty$, $\frac{1}{x} \to 0$. Since the tangent is a continuous function, $\tan(1/x) \to \tan(0) = 0$. Now, we're dividing this by x^2. That's the same as multiplying by $\frac{1}{x^2}$. We should know by now that

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

So, we've got $\tan(1/x) \to 0$ multiplied with $\frac{1}{x^2} \to 0$, hence the whole thing tends to zero. Thus, the limit is ZERO.

$$\lim_{x\to\infty} \sqrt{x^3+3x^2-9} - \sqrt{x^3-3x^2+9} \quad (1p)$$

We do the standard "completing the square" technique:

$$\sqrt{x^3+3x^2-9} - \sqrt{x^3-3x^2+9}$$
$$= \frac{(\sqrt{x^3+3x^2-9} - \sqrt{x^3-3x^2+9})(\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9})}{\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9}}$$
$$= \frac{x^3+3x^2-9-(x^3-3x^2+9)}{\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9}}$$
$$= \frac{6x^2}{\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9}}.$$

The denominator has dominant terms $x^{3/2}$. However, the numerator has a quadratic power. Therefore, the numerator is tending to infinity too fast for the denominator to keep up. Hence, the quotient is just getting bigger and bigger and bigger. If you'd like this made more precise, you can see that

$$\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9} \le 6\sqrt{x^3} \quad \forall x \ge 1000.$$

Hence

$$\frac{6x^2}{\sqrt{x^3+3x^2-9} + \sqrt{x^3-3x^2+9}} \ge \frac{6x^2}{6x^{3/2}} = \sqrt{x} \to \infty \text{ when } x \to \infty.$$

The limit therefore does NOT exist.

$$\lim_{x\to 0}\frac{e^x-1}{\sin(x)} \quad (1p)$$

We know that the derivative of e^x is e^x. We also know that $e^0 = 1$. So, by definition of derivative,

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x,$$

and in the special case that $x = 0$ we have

$$\lim_{h\to 0}\frac{e^h-1}{h} = e^0 = 1.$$

Changing the variable name, we see that

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1.$$

So, we can write

$$\lim_{x\to 0}\frac{e^x-1}{\sin(x)} = \lim_{x\to 0}\frac{e^x-1}{x}\frac{x}{\sin(x)} = 1.$$

Here we used the standard limit

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \implies \lim_{x\to 0}\frac{x}{\sin(x)} = 1.$$

Of course, you could always use Taylor series or l'hoptial's rule, those work fine too!

2. Bestäm om gränsvärdet finns och i så fall beräkna gränsvärdet: (1p)

$$\lim_{x\to 0}\left(1+\frac{x}{a}\right)^{2/x}$$

Yes, the limit exists. It is one of those e things. Let us change variables. We define

$$t = \frac{x}{a}.$$

Since a is presumably just a number, when $x \to 0$, $t \to 0$. So, the limit is

$$\lim_{t\to 0}(1+t)^{\frac{2}{ta}}.$$

One of the standard limits is that

$$\lim_{t\to 0}(1+t)^{1/t} = e.$$

By the continuity of the exponential function, we then have

$$\lim_{t\to 0}(1+t)^{\frac{1}{t}\frac{2}{a}} = e^{\frac{2}{a}}.$$

Hitta alla lösningar i $\mathbb{C}$ till ekvationen: $x^2 + 2x = -2$. (2p) We just use the quadratic formula. Re-writing the equation as

$$x^2 + 2x + 2,$$

the quadratic formula says the solutions are

$$\frac{-2 \pm \sqrt{4-4(2)}}{2} = -1 \pm \sqrt{-4}/2 = -1 \pm i.$$

3. Låt

$$f(x) = \frac{e^{\arctan(x)}}{x^2+1}.$$

(a) Beräkna $f'(x)$. (1p) It's all about the chain rule, and the product or quotient rule (they're equivalent :-) So, we use these to compute the derivative is:

$$f'(x) = \frac{\frac{e^{\arctan(x)}}{1+x^2}(x^2+1) - 2xe^{\arctan(x)}}{(x^2+1)^2}.$$

This simplifies to

$$f'(x) = \frac{e^{\arctan(x)}(1-2x)}{(x^2+1)^2}.$$

(b) Hitta alla extrempunkter av $f(x)$ i intervallet $[-1, 1]$. (2p) We look for the treasure points. First, we see that because e^{whatever} is never zero, the only way for $f'(x)$ to vanish is if $(1-2x) = 0$ which requires $x = \frac{1}{2}$. So, that's a treasure point. Since we're on a bounded interval, the endpoints -1 and 1 are also treasure points. We just need to compare what's happening. To be kind, it is enough to get full credit if you write: the smallest value amongst

$$\frac{e^{\arctan(-1)}}{2}, \quad \frac{e^{\arctan(1/2)}}{1/4+1}, \quad \frac{e^{\arctan(1)}}{2}$$

is the minimum on this interval. The largest value amongst these is the maximum. Of course, it is possible to think a bit more and recognize that $\arctan(-1) < \arctan(1/2)$, and $\arctan(-1) < \arctan(1)$. Therefore the minimum value is $\frac{e^{\arctan(-1)}}{2}$. The easiest thing to do, in my opinion, is to look at the sign of the derivative. When $x > \frac{1}{2}$ then $1 - 2x < 0$. Moreover, we can compute that $f'(x) < 0$ when $x > \frac{1}{2}$, and $f'(x) > 0$ when $x < \frac{1}{2}$. So, to the right of $1/2$, the function is decreasing, the graph is going down. To the left of $1/2$, the function is increasing, hence the graph is lower and going up towards $(1/2, f(1/2))$. Hence, we can see that $1/2$ is where the maximum point occurs, so the maximum value is $\frac{e^{\arctan(1/2)}}{1/4+1}$.

4. (Obs! 4Poäng!)

(a) Bevisa att derivatan av $\sin(x)$ är $\cos(x)$. (2p) (The proof is contained in the proof document on the course website!)

(b) Använd bara derivatans definition för att visa att derivatan av $\ln(x)$ är $\frac{1}{x}$. (2p)
We need to compute

$$\lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h}.$$

We use the rules for logarithms to see that

$$\ln(x+h) - \ln(x) = \ln\left(\frac{x+h}{x}\right) = \ln(1 + h/x).$$

So, we are computing

$$\lim_{h\to 0} \ln(1+h/x)(1/h).$$

This looks an awful lot like one of those e limits. Let's look instead at

$$\lim_{h\to 0} e^{\ln(1+h/x)(1/h)} = \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}.$$

Indeed, this is one of those e limits! Making the substitution $t = \frac{h}{x}$, then when $h \to 0$ we also have $t \to 0$. So, we compute

$$\lim_{t\to 0}(1+t)^{1/(tx)} = e^{1/x}.$$

Then, we have shown that

$$\lim_{h\to 0} e^{\ln(1+h/x)(1/h)} = e^{1/x}$$

which by continuity of ln shows that

$$\lim_{h\to 0} \ln(1+h/x)(1/h) = \frac{1}{x}.$$

Del 2

5. (a) Beräkna $\int_1^2 \frac{1}{e^x+1}dx$. (2 p.)

(b) Beräkna $\int_1^2 \ln(x)dx$. (1 p.)

$$\int_1^2 \frac{1}{e^x+1}dx = \int_1^2 \frac{e^x}{e^{2x}+e^x}dx = [y = e^x, dy = e^x dx] = \int_e^{e^2} \frac{1}{y^2+y}dy =$$

Partiellbråksuppdelning: $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$.

$$\int_e^{e^2} (\frac{1}{y} - \frac{1}{y+1})dy = [\ln(y) - \ln(y+1)]_e^{e^2} = \ln(e^2/(e^2+1)) - \ln(e/(e+1)) = \ln(e(e+1)/(e^2+1)).$$

$$\int_1^2 \ln(x)dx = [f(x) = \ln(x); g(x) = x] = [x\ln(x)]_1^2 - \int_1^2 x\frac{1}{x}dx = 2\ln(2) - 0 - \int_1^2 dx = 2\ln(2) - 1.$$

6. Ytan begränsad av kurvorna $y = e^x$, $y = x$, $x = 1$ och $x = 2$ roterar kring x-axeln. Beräkna rotationskroppens volym. (3 p.)

Vi kan se rotationskroppen som skilnad mellan ett man får om man roterar undergraf av e^x och ett man får om man roterar undergraf av x, begränsad av $x = 1$ och $x = 2$ i båda fall. Det första får vi som $\pi\int_1^2 (e^x)^2 dx = [y = 2x] = \pi/2 \int_2^4 e^y dy = \pi/2(e^4 - e^2)$ och det andra som $\pi\int_1^2 x^2 dx = \pi[\frac{1}{3}x^3]_1^2 = \pi\frac{1}{3}(8-1) = \pi\frac{7}{3}$. Så svaret är $\pi(\frac{1}{2}e^4 - \frac{1}{2}e^2 - \frac{7}{3})$.

7. (a) Lös differentialekvationen $y' = -x^2y + x^2$. (2 p.)

(b) Lös begynnelsevärdesproblemet $y' = -x^2y + x^2$, $y(0) = 2$. (1 p.)

a) Skriv om ekvationen som $y' + x^2y = x^2$. Detta är en första grads linjär ekvation. Så $\int x^2 dx = \frac{1}{3}x^3 + C$ och vi kan ta $G(x) = \frac{1}{3}x^3$. Multiplicering med $e^{G(x)} = e^{\frac{1}{3}x^3}$ ger: $(e^{\frac{1}{3}x^3}y)' = x^2e^{\frac{1}{3}x^3} = (e^{\frac{1}{3}x^3})'$. Alltså, $e^{\frac{1}{3}x^3}y = e^{\frac{1}{3}x^2} + C$, och $y = 1 + Ce^{-\frac{1}{3}x^3}$.

b) Från a) vi ser att $y = 1 + Ce^{-\frac{1}{3}x^3}$. Då $y(0) = 2 = 1 + Ce^{-\frac{1}{3}0^3} = 1 + C$, vi ser att $C = 1$. Svaret är $y = 1 + e^{-\frac{1}{3}x^3}$.

8. (a) Skriv Maclaurins formel till e^x med restterm i Lagrange form (för godtycklig n).(1 p.)

(b) Upskatta restterm i (a) för $x = 1$ och godtycklig n (man kan anta att $e < 3$) (1 p.)

(c) Hitta första siffran efter decimaltecken till e, med hjälp av (a) och (b) (1 p.)

a) $e^x = 1 + x/1! + x^2/2! + \ldots + x^n/n! + R_n$, där $R_n = e^{\theta x}x^{n+1}/(n+1)!$.

b) För $x = 1$ vi har $e^{\theta x}x^{n+1}/(n+1)! \leq e/(n+1)! \leq 3/(n+1)!$.

c) Ur b) för $n = 4$, $R_n \leq 3/120 = 1/40 < 0.05$, så $e^1 = 1 + 1 + 1/2 + 1/6 + 1/24 + R_n = 2.5 + 5/24 + R_n = 2.5 + 5/25 + (5/24 - 5/25) + R_n = 2.7 + 5(1/(25 \cdot 24)) + R_n$. Som vi har sett $R_n < 0.5$ och $5(1/(25 \cdot 24)) = 1/120 \leq 0.01$. Alltså $2.7 < e < 2.8$, i.e. första siffren efter decimaltecken är 7.

Lycka till!

Julie och Maria