Quantum Groups, Peter-Weyl Bases, Preferred Presentations and YBE

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June 7, 2019
Quantum Groups, Symmetric Spaces and Operator Algebras
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УДК 512.667.1 + 517.888 + 517.589

МАТЕМАТИКА

Л.Л. ВАКСМАН

q-АНАЛОГИ КОЭФФИЦИЕНТОВ КЛЕБША–ГОРДАНА И АЛГЕБРА ФУНКЦИЙ НА КВАНТОВОЙ ГРУППЕ $SU(2)$

(Представлено академиком И.М. Гельфандом 7 XII 1987)

1. В работе В. Хана [1] введены ортогональные многочлены, названные впоследствии q-полиномами Хана и малыми q-полиномами Якоби:

$$ {}_3\Phi_2\left[\begin{matrix} q^{-n}, & q^{n+1}ab, & q^{-x}; \; q, q \\ q^a, & q^{-N} & \end{matrix}\right], \qquad {}_2\Phi_1\left[\begin{matrix} q^{-n}, q^{n+1}\alpha\beta; \; q, \; qx \\ q^{\alpha} \end{matrix}\right]. $$

Здесь ${}_r\Phi_s$ – базисный гипергеометрический ряд (см. [2, стр. 196]).

В настоящей работе показано, каким образом эти полиномы возникают в связи с теорией представлений квантовой группы $SU(2)$.

Как известно [3], при $q = 1$ полиномы Хана несущественно отличаются от коэффициентов Клебша–Гордана. Основными результатами настоящей работы являются q-аналоги формулы Рака–Фока для этих коэффициентов и производящей функции для них. (Результаты об обычных коэффициентах Клебша–Гордана см. в [4].)

2. Основые теории квантовых групп изложены в обзорах В.Г. Дринфельда [5] и М. Дзимбо [6]. Пусть $q \in \mathbf{R}$, $q > 0$, $q \neq 1$. Рассмотрим алгебру Хопфа $U_q\,\mathrm{sl}(2)$, определенную образующими X, Y, E_+, E_- и соотношениями

$$ XY - YX = (E_+^2 - E_-^2)/(q^{1/2} - q^{-1/2}), \quad E_\pm X = q^{\pm 1/2} XE_\pm, $$
$$ E_\pm Y = q^{\mp 1/2} YE_\pm, \quad E_\pm \cdot E_\mp = E_- \cdot E_+ = 1, \quad \Delta(X) = X \otimes E_+ + E_- \otimes X, $$
$$ \Delta(Y) = Y \otimes E_+ + E_- \otimes Y, \qquad \Delta(E_\pm) = E_\pm \otimes E_\pm. $$

Vaksman, L. L.(2-ROST)

q-analogues of Clebsch-Gordan coefficients, and the algebra of functions on the quantum group $\mathrm{SU}(2)$. (Russian)

Dokl. Akad. Nauk SSSR **306** (1989), no. 2, 269--271

Formal deformation of bialgebras

Let B be a $\mathbb{C}$-bialgebra. Consider a formal bialgebra deformation $B_\hbar$, i,e, a $\mathbb{C}[[\hbar]]$-bialgebra such that

- $B_\hbar \simeq B[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules.
- There is a bialgebra isomorphism $B_\hbar \simeq B \mod \hbar$.

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Thus $\mu(x, y) = \mu_0(x, y) + \hbar\mu_1(x, y) + \hbar^2\mu_2(x, y) + \cdots$ and

$$\Delta(x) = \Delta_0(x) + \hbar\Delta_1(x) + \hbar\Delta_2(x) + \cdots$$

where μ_0, Δ_0 are the original multiplication and comultiplication. Formal bialgebra deformations controlled by *Gerstenhaber-Schack Cohomology* with usual obstruction theory.

Fundamental Example: $\mathcal{O}_\hbar(M(2))$ and $\mathcal{O}_\hbar(SL(2))$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the coordinate functions on $M_{2\times 2}$ and (usually) set $q = \exp(\hbar)$. The quantization is given by $\mathbb{C}\langle a, b, c, d\rangle$ with the relations

- $ab = q\, ba$
- $ac = q\, bc$
- $bd = q\, db$
- $cd = q\, dc$
- $ad - da = (q - q^{-1})bc$
- $bc = cb$
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The comultiplication is given by

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

on generators and extended to be an algebra map.

Rigidity Results

Let G be a reductive algebraic group or monoid.

- The Hochschild $H^n(U(\mathfrak{g}), U(\mathfrak{g})) = 0$ $(n \geq 0)$ so $U(\mathfrak{g})$ is rigid as an algebra.
- The coalgebra $H^n(\mathcal{O}(G), \mathcal{O}(G)) = 0$ $(n \geq 0)$ so $\mathcal{O}(G)$ is rigid as coalgebra.

Thus every deformation of $U(\mathfrak{g})$ is equivalent to one in which $\mu(x, y) = \mu_0(x, y)$ and every deformation of $\mathcal{O}(G)$ is equivalent to one in which $\Delta(x) = \Delta_0(x)$.

Preferred Presentations

Equivalent bialgebra deformations of $U(\mathfrak{g})$ and $\mathcal{O}(G)$ with strictly unchanged multiplication or comultiplication on all elements are called **preferred presentations**.

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- $\Delta(a^2) = a^2 \otimes a^2 + (1 + e^{-2\hbar}) ab \otimes ac + b^2 \otimes c^2$ in $\mathcal{O}_\hbar(SL(2))$. (Using PBW basis.)

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Problem: Find preferred presentations of $U_\hbar(\mathfrak{g})$ and $\mathcal{O}_\hbar(G)$. That is, find $\Delta_\hbar$ and $\mu_\hbar$ such that

- $(U(\mathfrak{g})[[\hbar]], \mu_0, \Delta_\hbar) \simeq U_\hbar(\mathfrak{g})$
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A common approach in quantization is to identify the classical limit or infinitesimal deformations. In many nice cases, all of these can be integrated to a full deformation. [Kontsevich, Etingof-Kazhdan, PBW-type/Koszul deformations].

For $B = \mathcal{O}(G)$ or $U(\mathfrak{g})$ the classical limits are solutions to the (M)CYBE in $\mathfrak{g} \otimes \mathfrak{g}$.

Infinitesimal Deformations by Example

Consider $\mathrm{GL}(2)$ and $\mathfrak{g} = \mathfrak{gl}_2$ with basis

$$e = e_{12}, \; f = e_{21}, \; h_1 = e_{11}, \; h_2 = e_{22}.$$

There are three classical r-matricees up to equivalence.

- Moyal type: $h_1 \wedge h_2$ Triangular and quasi-Frobenius
- Jordan type: $h_1 \wedge e$ Triangular and Frobenius
- Drinfel'd-Jimbo type: $e \wedge f$ (quasi-triangular)

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Triangular Case – Universal Deformation Formulae – Quantum Cocycles

- Moyal – Exponential: $\exp(\hbar(h_1 \otimes h_2))$

$$1 \otimes 1 + \hbar\ h_1 \otimes h_2 + \frac{\hbar^2}{2!} h_1^2 \otimes h_2^2 + \frac{\hbar^3}{3!} h_1^3 \otimes h_2^3 + \quad \cdots$$

- Jordan – Quasi-Exponential (Coll-Gerstenhaber-Giaquinto 1989)

$$1 \otimes 1 + \hbar\ h_1 \otimes e + \frac{\hbar^2}{2} h(h+1) \otimes e^2 + \frac{\hbar^3}{3!} h(h+1)(h+2) \otimes e^3 + \cdots$$

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- Drinfel'd-Jimbo type: $\exp(\hbar\ e \otimes f)$ gives correct relations, but is not associative or quantum cocycle. Perhaps it is a dual quasi-bialgebra?

Interesting failed attempt for Drinfel'd-Jimbo types

Consider the skew biderivation of $\mathcal{O}(M_{2\times 2})$ corresponding to $e \wedge f$:

$r = (a\partial_b + c\partial_d) \wedge (b\partial_a + d\partial_c) - (c\partial_a + d\partial_b) \wedge (a\partial_c + b\partial_d)$.

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This gives the standard Poisson-Lie structure on $\mathcal{O}(M_{2\times 2})$.

For $\alpha, \beta \in \mathcal{O}(M_{2\times 2})$ define

$$\alpha * \beta = \mu_0(\exp \hbar\, r)(\alpha \otimes \beta).$$

Then the commutation relations of the generators a, b, c, d coincide exactly with those of $\mathcal{O}_\hbar(M_{2\times 2})$ and

$$\Delta_0(\alpha * \beta) = (\Delta_0(\alpha))(\Delta_0(\beta)).$$

Is the above part of a dual quasi-bialgebra structure?

The Peter-Weyl approach

The *Peter-Weyl* decomposition of $\mathcal{O}_\hbar(G)$ yields a preferred presentation. The idea comes from a dual point of view.

$$\mathcal{O}_\hbar(G) = (U_\hbar(\mathfrak{g}))^\circ \simeq \bigoplus_\lambda \mathrm{End}(V_\lambda)$$

where λ runs through the finite dim'l irreducible representations of $U_\hbar(\mathfrak{g})$. The last equivalence is valid as $\mathcal{O}_\hbar(G)$ is cosemisimple, and all V_λ correspond to irreducible representations of $\mathcal{O}(G)$ when $\hbar = 0$.

Peter-Weyl Basis

The Peter-Weyl decomposition gives a natural way to choose a basis for $\mathcal{O}_\hbar(G)$: pick dual bases B_λ and B_λ^* for each V_λ and its dual V_λ^*. Then

$$\bigsqcup_\lambda \{X \otimes Y^* : X, Y \in B_\lambda\}. \tag{1}$$

is a basis for $\mathcal{O}_\hbar(G)$, which we call a **Peter-Weyl basis**.

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Comultiplication In both $\mathcal{O}(G)$ and $\mathcal{O}_\hbar(G)$, comultiplication is the dual of multiplication on $\oplus_\lambda \mathrm{End} V_\lambda$. In coordinates that means that, for $X, Y \in B_\lambda$,

$$\Delta(X \otimes Y^*) = \sum_{Z \in B_\lambda} (X \otimes Z^*) \otimes (Z \otimes Y^*). \tag{2}$$

3j Symbols (Clebsch-Gordan Coefficients)

For each triple λ, μ, ν, there is a decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu$$

Thus we have two bases for $V_\lambda \otimes V_\mu$

$$\{X \otimes Y \mid X \in B_\lambda,\ Y \in B_\mu\} \quad \text{and} \quad \{Z \mid Z \in B_\nu\}$$

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For each $X \in B_\lambda, Y \in B_\mu, Z \in B_\nu$, write

$$X \otimes Y = \sum_\nu \sum_{Z \in B_\nu} \begin{pmatrix} \lambda & \mu & \nu \\ X & Y & Z \end{pmatrix} Z. \tag{3}$$

The constants $\begin{pmatrix} \lambda & \mu & \nu \\ X & Y & Z \end{pmatrix} \in \mathbb{C}[[\hbar]]$ are called the 3j symbols.

3j Symbols are Complicated!

For $\mathfrak{sl}_2$, the 3j Symbol $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$ equals the following:

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For $\mathfrak{sl}_2$, the 3j Symbol $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$ equals the following:

$$= (-1)^{j_1-m_1} q^{j_2(j_2+1)-j_1(j_1+1)-j(j+1)+2m_1(m+1)} \times$$

$$\times \left\{ \frac{[j+m]![j-m]![j_1-m_1]![j_2-m_2]![j_1+j_2-j]![2j+1]_1}{[j_1+m_1]![j_2+m_2]![j_1-j_2+j]![j_2-j_1+j]![j_1+j_2+j+1]!} \right\}^{1/2} \times \qquad \text{(I7)}$$

$$\sum_{z \geqslant 0} (-1)^z q^{2z(m+j+1)} \frac{[j_1+m_1+z]![j_2+j-m_1-z]!}{[z]![j-m-z]![j_1-m_1-z]![j_2-j_1+m_1+z]!}$$

Structure Constants

The structure constants for multiplication of Peter-Weyl basis elements can be explicitly determined. For $X_1, Y_1 \in B_\lambda$ and $X_2, Y_2 \in B_\mu$,

$$(X_1 \otimes Y_1^*) *_\hbar (X_2 \otimes Y_2^*) = \sum_{\substack{\nu \\ X_3, Y_3 \in B_\nu}} \begin{pmatrix} \lambda & \mu & \nu \\ X_1 & X_2 & X_3 \end{pmatrix} \begin{pmatrix} \lambda & \mu & \nu \\ Y_1^* & Y_2^* & Y_3^* \end{pmatrix} X_3 \otimes Y_3^*.$$

The Preferred Presentation

Theorem

Let $\mathcal{O}_\hbar(G)$ be the standard Drinfel'd-Jimbo quantum function Hopf algebra, and let $*_h$ denote the multiplication given by the 3j symbols. Then

$$(\mathcal{O}(G)[[\hbar]], *_h, \Delta_0) \cong \mathcal{O}_\hbar(G)$$

and thus is a preferred presentaion.

Belavin-Drinfeld Classification

Let $\mathfrak{g}$ be a simple Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Belavin and Drinfeld classified all solutions to CYBE

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

satisfying $r + r_{21} = t$, the Casimir element.
The space of solutions has a discrete combinatorial parameter (triple) and a continuous parameter associated to each triple.

BD Triples

Let Γ be the set of positive roots. The discrete parameter consists of a **BD triple** $(\Gamma_1, \Gamma_2, \tau)$ with $\Gamma_i \subset \Gamma$ and $\tau : \Gamma_1 \to \Gamma_2$ a bijection satisfying

1. $(\tau\alpha, \tau\beta) = (\alpha, \beta)$ for all $\alpha, \beta \in \Gamma$;
2. For every $\alpha \in \Gamma_1$, there is a $k \geq 0$ with $\tau^k\alpha \in \Gamma_1$ but $\tau^{k+1}\alpha \notin \Gamma_1$.

For $\alpha \in \tilde{\Gamma_1}$, $\beta \in \tilde{\Gamma_2}$, define $\alpha \prec \beta$ if $\tau^k(\alpha) = \beta$.
Associated to each triple is a $\binom{d}{2}$ dimensional continuous parameter space of $(\mathfrak{h} \otimes \mathfrak{h})_\tau$ of dimension $d = \#(\Gamma - \Gamma_1)$.

Belavin and Drinfeld showed that every solution is equivalent to one of the form

$$r = r^0 + \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge e_\beta$$

for some triple τ and $r^0 \in (\mathfrak{h} \otimes \mathfrak{h})_\tau$

Standard DJ triple: $\Gamma_1 = \Gamma_2 = \emptyset$. Here $\dim(\mathfrak{h} \otimes \mathfrak{h})_\tau = \binom{n-1}{2}$. This is the largest component of solutions.

Some interesting triples in $\mathfrak{sl}_n$

- Cremmer-Gervais triple:

  $$\Gamma_1 = \{1, 2, \ldots, n-2\} \quad \Gamma_2 = \{2, 3, \ldots, n-1\}, \quad \tau(i) = i + 1.$$

  Here r_0 is unique.

- Generalized Cremmer-Gervais: Suppose i and n are relatively prime.

  $$\Gamma_1 = \Gamma \setminus \{n-i\} \quad \Gamma_2 = \Gamma \setminus \{i\} \quad \tau(j) = j + i \mod n.$$

  Here too r^0 is unique.

Question: Do GCG r-matrices degenerate into Frobenius unitary r-matrices for maximal parabolic subalgebra associated to root i? True for CG $i = 1$ and all n.

The GGS Conjecture/Theorem

Let $a = \sum_{\substack{\alpha,\beta>0 \\ \alpha\prec\beta}} e_{-\alpha} \wedge e_\beta$ and $c = \sum_{\alpha>0} e_{-\alpha} \wedge e_\alpha$.

Set $\epsilon = -(ac + ca + a^2)$. Now define $\tilde{a}$ by $\tilde{a} = \sum a_{jl}^{ik} q^{a_{jl}^{ik}\epsilon_{jl}^{ik}} e_{ij} \otimes e_{kl}$ where $a = \sum a_{jl}^{ik} e_{ij} \otimes e_{kl}$ and similarly for ϵ. Set $\hat{q} = q - q^{-1}$.

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Theorem

Let τ be a Belavin-Drinfeld triple for $\mathfrak{sl}_n$ and suppose $r^0 = t^0/2 + \tilde{r}^0$ is τ-admissible. Then the matrix

$$R = q^{\tilde{r}^0} (R_s + \hat{q}\,\tilde{a})\, q^{\tilde{r}^0}$$

satisfies the quantum Yang-Baxter equation and PR satisfies the Hecke relation.

Generalized Cremmer-Gervais R-Matrix for n=5 and i=2

$$
\begin{pmatrix}
q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{\frac{1}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-\frac{3}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{\frac{3}{5}} & 0 & 0 & 0 & \hat{p}q^{\frac{2}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-\frac{1}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{p}q^{\frac{2}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{q} & 0 & 0 & 0 & q^{-\frac{1}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{q}q^{-\frac{4}{5}} & 0 & 0 & 0 & q & 0 & 0 & 0 & \hat{p}q^{\frac{4}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q}q^{-\frac{4}{5}} & 0 & 0 & 0 & q & 0 & 0 & 0 & \hat{p}q^{\frac{4}{5}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{5}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \hat{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q}q^{-\frac{2}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{1}{5}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q}q^{-\frac{2}{5}} & 0 & 0 & 0 & q^{-\frac{3}{5}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{\frac{3}{5}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{q} & 0 & 0 & 0 & q^{-\frac{1}{5}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q
\end{pmatrix}.
$$

Thank you!