q-INDEPENDENCE OF THE DRINFIELD-JIMBO QUANTIZATION

Olof Giselsson

June 4, 2019

INTRODUCTION

Let G be a connected simply connected compact Lie group and let $q \in (0,1)$.

Let $\mathbb{C}[\mathrm{G}]_{\mathrm{q}}$ be the Drinfeld-Jimbo quantization of $\mathbb{C}[\mathrm{G}] = \mathbb{C}[\mathrm{G}]_1$.

- $\mathbb{C}[\mathrm{G}]_{\mathrm{q}}$ is a $*$-Hopf algebra,
- is a q-analogue of regular functions on G.

Let $C(\mathrm{G})_{\mathrm{q}}$ be the universal enveloping C^*-algebra of $\mathbb{C}[\mathrm{G}]_{\mathrm{q}}$.

- $C(\mathrm{G})_{\mathrm{q}}$ is a compact quantum group.
- As a C^*-algebra it is Type I (every irreducible $*$-representation contains the compact operators).

POISSON-LIE GROUP

For every $f \in \mathbb{C}[\mathrm{G}]$ we have a map $q \in (0,1] \mapsto f(q) \in \mathbb{C}[\mathrm{G}]_{\mathrm{q}}$, such that

- we have $f(1) = f$,
- for each $q \in (0,1)$ we have a linear isomorphism $f \in \mathbb{C}[\mathrm{G}] \mapsto f(q) \in \mathbb{C}[\mathrm{G}]_{\mathrm{q}}$,
- we have a Poisson bracket on $\mathbb{C}[\mathrm{G}]$ induced by the quantization

$$[f,g] := \lim_{q \to 1} \frac{1}{1-q}(f(q)g(q) - g(q)f(q))$$

i.e. a Lie bracket on $\mathbb{C}[\mathrm{G}]$ such that $[fg,h] = f[g,h] + [f,h]g$, $\forall f,g,h \in \mathbb{C}[\mathrm{G}]$.

Symplectic Leaves

This gives a Poisson-Lie group structure on G.
Decomposes G as a disjoint union of minimal symplectic manifolds U called its *symplectic leaves*.

- Each symplectic leaf U is of even dimension.
- Are parameterized by (σ, t) for σ in the Weyl group W and t in the maximal torus T of G.
- $\dim \mathrm{U}_{(\sigma,t)} = 2 \times \text{length}(\sigma)$.
- $\mathrm{U}_{(s,1)} \subseteq \overline{\mathrm{U}}_{(\sigma,1)}$ if and only if $s \leq \sigma$ in the partial order of W.

Example

- If $\mathrm{G} = SU(2)$ then $\mathrm{W} = S_2 = \{e, s_1\}$ and $\mathrm{T} = \mathbb{T}$.
- If $\mathrm{G} = SU(3)$ then $\mathrm{W} = S_3 = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ and $\mathrm{T} = \mathbb{T}^2$.

Representation Theory of $C(\mathrm{G})_\mathrm{q}$

Theorem (Soibelman)

There is a one-to-one correspondence between equivalence classes of irreducible $*$-representations of $C(\mathrm{G})_\mathrm{q}$ and symplectic leaves of G.

- $\pi_{\mathrm{U}}^{(q)}$ irreducible $*$-representation $\quad\leftrightarrow\quad$ U symplectic leaf
- We have $\mathrm{U} \subseteq \overline{\mathrm{V}}$ if and only if $\ker \pi_{\mathrm{V}}^{(q)} \subseteq \ker \pi_{\mathrm{U}}^{(q)}$,
- There is a Hilbert space H_{U}, such that $\pi_{\mathrm{U}}^{(q)} : C(\mathrm{G})_\mathrm{q} \to \mathcal{B}(H_{\mathrm{U}})$ and

$$q \in (0,1) \mapsto \pi_{\mathrm{U}}^{(q)}(f(q)) \in \mathcal{B}(H_{\mathrm{U}})$$

  is continuous for all $f \in \mathbb{C}[\mathrm{G}]$.

Statement of Result

Let us define $C(\overline{\mathrm{U}})_{\mathrm{q}} := \pi_{\mathrm{U}}^{(q)}(C(\mathrm{G})_{\mathrm{q}}) \subseteq \mathcal{B}(H_{\mathrm{U}})$. We state the main result:

Theorem (O. Giselsson)

For a fixed symplectic leaf $\mathrm{U} \subseteq \mathrm{G}$, we have an inner isomorphism

$$C(\overline{\mathrm{U}})_{\mathrm{q}} \cong C(\overline{\mathrm{U}})_{\mathrm{p}} \quad \text{for all } q, p \in (0, 1).$$

Moreover $C(\mathrm{G})_{\mathrm{q}} \cong C(\mathrm{G})_{\mathrm{p}}$ as C^*-algebras for all $q, p \in (0, 1)$.

Previously Known Cases

Example ($\mathrm{G} = SU(2)$)

This was shown by Woronowicz. The isomorphism $C(SU(2))_q \cong C(SU(2))_p$ can be explicitly constructed.
In particular

$$C(\overline{\mathrm{U}}_{(s_1,t)})_q \cong C(\overline{\mathrm{U}}_{(s_1,t)})_p \cong \mathcal{T},$$

where $\mathcal{T}$ is the Toeplitz algebra.

Example ($\mathrm{G} = SU(3)$)

This was proven by G. Nagy in the 90's. No longer an explicit isomorphism. Works by "untwisting" the $*$-representation corresponding to $\mathrm{U} := \mathrm{U}_{(s_1 s_2 s_1, 1)}$. Then apply a lifting lemma.

Outline of Nagy's Proof

Let $\pi^{(q)}$ correspond to U (the big symplectic leaf). Note that $\mathcal{K}(H_{\mathrm{U}}) \subseteq C(\overline{\mathrm{U}})_{\mathrm{q}}$. Consider the $*$-representation

$$\Psi^{(q)} : C(SU(3)) \stackrel{\pi^{(q)}}{\rightarrow} C(\overline{\mathrm{U}})_{\mathrm{q}} \stackrel{P}{\rightarrow} C(\overline{\mathrm{U}})_{\mathrm{q}}/\mathcal{K}(H_{\mathrm{U}}) \subseteq \mathcal{Q}(H_{\mathrm{U}}).$$

(P is the quotient map).

- Let $\Pi_1^{(q)}$ be the direct integral of $*$-representations corresponding to $(s_1 s_2, t)$ for $t \in \mathrm{T}$
- Let $\Pi_2^{(q)}$ be the direct integral of $*$-representations corresponding to $(s_2 s_1, t)$ for $t \in \mathrm{T}$

Outline of Nagy's Proof

There is an isomorphism

$$\varphi_{(q)} : C(\overline{\mathrm{U}})_{\mathrm{q}}/\mathcal{K}(H_{\mathrm{U}}) \to \mathrm{Im}(\Pi_1^{(q)} \oplus \Pi_2^{(q)}),$$

$$\varphi_{(q)} \circ \Psi^{(q)} = \Pi_1^{(q)} \oplus \Pi_2^{(q)}.$$

Moreover,

$$\mathrm{Im}(\Pi_1^{(q)} \oplus \Pi_2^{(q)}) = \mathrm{Im}(\Pi_1^{(p)} \oplus \Pi_2^{(p)}), \quad \text{for all } q, p \in (0, 1)$$

This can be explicitly seen!

Lifting

Denote $M := \mathrm{Im}(\Pi_1^{(q)} \oplus \Pi_2^{(q)})$. The maps

$$\varphi_{(q)}^{-1} : M \to \mathcal{Q}(H_{\mathrm{U}}), \quad q \in (0,1)$$

can be shown to vary continuously point-wise , i.e

$$q \in (0,1) \mapsto \varphi_{(q)}^{-1}(a)$$

is continuous for all $a \in M$. Moreover $P^{-1}(\varphi_{(q)}^{-1}(M)) = C(\overline{\mathrm{U}})_{\mathrm{q}}$.

NAGY'S LIFTING LEMMA.

LEMMA (G. NAGY)

Let A be a type I C^*-algebra, H a separable Hilbert space, $Q : \mathcal{B}(H) \to \mathcal{Q}(H)$ the quotient map and

$$A \xrightarrow{\phi_t} \mathcal{Q}(H), \quad t \in [0,1]$$

are injective homomorphisms s.t. $\forall a \in A$, the map $t \in [0,1] \mapsto \phi_t(a)$ is continuous, then

$$Q^{-1}(\phi_t(A)) \cong Q^{-1}(\phi_s(A)), \quad \forall t, s \in [0,1].$$

NAGY'S PROOF CONT.

Using this lemma, Nagy proved

$$C(\overline{\mathrm{U}})_{\mathrm{q}} = P^{-1}(\varphi^{-1}_{(q)}(M)) \cong P^{-1}(\varphi^{-1}_{(p)}(M)) = C(\overline{\mathrm{U}})_{\mathrm{p}}, \quad \text{for all } q, p \in (0, 1)$$

and then $C(SU(3))_q \cong C(SU(3))_p$.
These isomorphisms are inner i.e. are implemented by a unitary isometry.

General Proof

We use induction on the dimension $2m$ of the symplectic leaf.
If $\dim \mathrm{U} = 0$, then $\pi_{\mathrm{U}}^{(q)}$ is a $*$-homomorphism to $\mathbb{C}$.
Hence
$$C(\overline{\mathrm{U}})_q = \mathbb{C}.$$
To explain $\dim \mathrm{U} > 0$, we will introduce a "commutative model".

Commutative Model

If $\dim \mathrm{U} = 2m > 0$, then we have a disjoint union

$$\overline{\mathrm{U}} = \mathrm{U} \cup \left(\cup_j \mathrm{U}_j^{(m-1)}\right) \cup \left(\cup_j \mathrm{U}_j^{(m-2)}\right) \cup \cdots \cup \left(\cup_j \mathrm{U}_j^{(0)}\right)$$

over symplectic leaves $\subseteq \overline{\mathrm{U}}$, where $\dim \mathrm{U}_j^{(k)} = 2k$.
In particular

$$\overline{\mathrm{U}} = \mathrm{U} \cup \left(\cup_j \overline{\mathrm{U}}_j^{(m-1)}\right)$$

gives a $*$-homomorphism

$$C(\overline{\mathrm{U}}) \stackrel{\partial_{m-1}}{\to} \prod_j C(\overline{\mathrm{U}}_j^{(m-1)})$$

with $\ker \partial_{m-1} = C_0(U)$.

Commutative Model cont.

We can iterate this to get a sequence

$$C(\overline{\mathrm{U}}) \stackrel{\partial_{m-1}}{\rightarrow} \prod_j C(\overline{\mathrm{U}}_j^{(m-1)}) \stackrel{\partial_{m-2}}{\rightarrow} \prod_j C(\overline{\mathrm{U}}_j^{(m-2)}) \stackrel{\partial_{m-2}}{\rightarrow} \cdots$$

$$\cdots \stackrel{\partial_1}{\rightarrow} \prod_j C(\overline{\mathrm{U}}_j^{(1)}) \stackrel{\partial_0}{\rightarrow} \prod_j C(\overline{\mathrm{U}}_j^{(0)})$$

s.t $\ker \partial_k = \prod_j C_0(\mathrm{U}_j^{(k+1)})$

q-Analogue of Commutative Model

Setting $C_0(\mathrm{U})_q := \mathcal{K}(H_{\mathrm{U}})$, we have a q-analogue

$$C(\overline{\mathrm{U}})_q \overset{\partial^{(q)}_{m-1}}{\to} \prod_j C(\overline{\mathrm{U}}_j^{(m-1)})_q \overset{\partial^{(q)}_{m-2}}{\to} \prod_j C(\overline{\mathrm{U}}_j^{(m-2)})_q \overset{\partial^{(q)}_{m-2}}{\to} \cdots$$

$$\cdots \overset{\partial^{(q)}_{1}}{\to} \prod_j C(\overline{\mathrm{U}}_j^{(1)})_q \overset{\partial^{(q)}_{0}}{\to} \prod_j C(\overline{\mathrm{U}}_j^{(0)})_q$$

s.t $\ker \partial_k^{(q)} = \prod_j C_0(\mathrm{U}_j^{(k+1)})_q$.
This can be deduced from the representation theory of $C(\mathrm{G})_{\mathrm{q}}$.

A LADDER

By induction on m, we have $*$-isomorphisms $\Gamma_k^{p,q}$ s.t

$$\begin{array}{ccccccccc}
C(\overline{\mathrm{U}})_q & \xrightarrow{\partial_{m-1}^{(q)}} & \prod_j C(\overline{\mathrm{U}}_j^{(m-1)})_q & \xrightarrow{\partial_{m-2}^{(q)}} & \cdots & \xrightarrow{\partial_1^{(q)}} & \prod_j C(\overline{\mathrm{U}}_j^{(1)})_q & \xrightarrow{\partial_0^{(q)}} & \prod_j C(\overline{\mathrm{U}}_j^{(0)})_q \\
\downarrow & & \downarrow \Gamma_{m-1}^{p,q} & & & & \downarrow \Gamma_1^{p,q} & & \downarrow \Gamma_0^{p,q} \\
C(\overline{\mathrm{U}})_p & \xrightarrow{\partial_{m-1}^{(p)}} & \prod_j C(\overline{\mathrm{U}}_j^{(m-1)})_p & \xrightarrow{\partial_{m-2}^{(p)}} & \cdots & \xrightarrow{\partial_1^{(p)}} & \prod_j C(\overline{\mathrm{U}}_j^{(1)})_p & \xrightarrow{\partial_0^{(p)}} & \prod_j C(\overline{\mathrm{U}}_j^{(0)})_p
\end{array}$$

is commutative. We want to construct the dotted arrow.
Let $C(\partial^k\overline{\mathrm{U}})_q$ be the image of $C(\overline{\mathrm{U}})_q$ in $\prod_j C(\overline{\mathrm{U}}_j^{(k)})_q$.
We want to prove

$$\Gamma_k^{p,q}(C(\partial^k\overline{\mathrm{U}})_q) = C(\partial^k\overline{\mathrm{U}})_p \quad \text{for } k = 0, 1, \ldots, m-1.$$

We do this by induction on k. It is trivial for $k = 0$.

Outline of Arguments

We have results:

(i) the C^*-algebras $C(\partial^{(0)}\overline{\mathrm{U}})_q$ are commutative and isomorphic for all $q \in (0,1)$, via $\Gamma_0^{p,q}$,

(ii) for $k = 0, \ldots, m-1$ the intersection of $C(\partial^{(k)}\overline{\mathrm{U}})_q$ with $\prod C_0(\mathrm{U}_j^{(k)})_q$ is mapped by $\Gamma_k^{p,q}$ to the intersection of $C(\partial^{(k)}\overline{\mathrm{U}})_p$ with $\prod C_0(\mathrm{U}_j^{(k)})_p$,

(iii) for $k = 1, \ldots, m-1$, there is an approximate unit $\{u_{q,i}^{(k)}\}_{i=1}^{\infty}$ for $\prod_j C_0(\mathrm{U}_j^{(k)})_q$ such that $\{u_{q,i}^{(k)}\}_{i=1}^{\infty} \subseteq C(\partial^{(k)}\overline{\mathrm{U}})_q$ and $\Gamma_k^{p,q}(u_{q,i}^{(k)}) = u_{p,i}^{(k)}$ for all $i \in \mathbb{N}$.

Outline of Arguments Cont.

These results, combined with the commutivity of the diagram

$$\begin{array}{ccc}
\prod_j C(\overline{\mathrm{U}}_j^{(k)})_q & \xrightarrow{\partial_{k-1}^{(q)}} & \prod_j C(\overline{\mathrm{U}}_j^{(k-1)})_q \\
\Gamma_k^{p,q} \downarrow & & \Gamma_{k-1}^{p,q} \downarrow \\
\prod_j C(\overline{U}_j^{(k)})_p & \xrightarrow{\partial_{k-1}^{(p)}} & \prod_j C(\overline{U}_j^{(k-1)})_p
\end{array}$$

gives that

$$\Gamma_{k-1}^{p,q}(C(\partial^{k-1}\overline{\mathrm{U}})_q) = C(\partial^{k-1}\overline{\mathrm{U}})_p$$

implies

$$\Gamma_k^{p,q}(C(\partial^k\overline{\mathrm{U}})_q) = C(\partial^k\overline{\mathrm{U}})_p$$

Outline of Arguments Cont.

In particular, we have isomorphisms

$$C(\overline{\mathrm{U}})_q/C_0(\mathrm{U})_q \cong C(\partial^{m-1}\overline{\mathrm{U}})_q \overset{\Gamma^{p,q}_{m-1}}{\to} C(\partial^{m-1}\overline{\mathrm{U}})_p \cong C(\overline{\mathrm{U}})_p/C_0(\mathrm{U})_p.$$

After checking that

$$C(\overline{\mathrm{U}})_q/C_0(\mathrm{U})_q \cong C(\overline{\mathrm{U}})_p/C_0(\mathrm{U})_p$$

varies continuously on $q, p \in (0, 1)$, we use a modified version of Nagy's lemma to show

$$C(\overline{\mathrm{U}})_q \cong C(\overline{\mathrm{U}})_p.$$

THANK YOU!