Quantized Differential Operators, Quantized Weyl Algebras, and Quantized Generalized Verma Modules

In memory of Leonid Vaksman. Kristineberg 2-8 June 2019

Hans Plesner Jakobsen
University of Copenhagen

June 5 2019

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$Su(n, n) \subseteq Sp(n, \mathbb{R})$ (look at $Im(\cdot, \cdot)_{n,n}$)

The Kashiwara-Vergne Conjecture (KVC)

KVC: One gets all unitary highest weight representations of $SU(n,n)$ from tensor products of the (not irreducible) Harmonic representation.

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Idea of the proof of the KVC and ideas therefrom: The Harmonic representation contains "fewer" K-types than the generic unitary h.w. representations. So there must be a proper quotient. On the most singular K-types, the Hermitian form is a first order polynomial in λ (from the center of K)

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$\Lambda \leftrightarrow (\Lambda_0, \lambda)$.
The generalized Verma module $M(V_\Lambda)$:

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\mathcal{M}(V_\Lambda) = \mathcal{U}(\mathfrak{g}^{\mathbb{C}}) \bigotimes_{\mathcal{U}(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+)} V_\Lambda \tag{1}
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Specifically, as a vector space, $M(V_{\tau'})$ ca be seen as the space of (constant coefficient) $V_{\tau'}$-valued holomorphic differential operators on $\mathfrak{p}^-$. (Domain $G/K = \mathcal{D} \subseteq \mathfrak{p}^-$.)

$$(q, p) = (q(\frac{\partial}{\partial Z})p)(0).$$

Homomorphisms

$$\phi : M(V_{\tau_1'}) \to M(V_{\tau'}) \text{ homomorphism}$$

determined as follows: $\{f_1, f_2, \ldots, f_{N_1}\}$ and $\{e_1, e_2, \ldots, e_N\}$ bases of $V_{\tau_1'}$ and $V_{\tau'}$, respectively. Define the $N \times N_1$ matrix $\mathbf{\Phi}$ as follows:

$$\mathbf{\Phi}_{ij} = \phi(f)_{ith \text{ coord.}} \in \mathcal{U}(\mathfrak{p}^-).$$

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$$M(V_{\tau_1'}) \ni \begin{pmatrix} q_1 \\ \vdots \\ q \\ \vdots \\ q_{N_1} \end{pmatrix} \to \begin{pmatrix} \mathbf{\Phi}_{11} & \ldots & \mathbf{\Phi}_{1j} & \ldots & \mathbf{\Phi}_{1N_1} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \mathbf{\Phi}_{i1} & \ldots & \mathbf{\Phi}_{ij} & \ldots & \mathbf{\Phi}_{iN_1} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ \mathbf{\Phi}_{N1} & \ldots & \mathbf{\Phi}_{Nj} & \ldots & \mathbf{\Phi}_{NN_1} \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q \\ \vdots \\ q_{N_1} \end{pmatrix}$$

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Theorem. There is a simple bijective correspondence between covariant differential operators and homomorphisms between generalized Verma modules.

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$$\mathcal{H} = Span_{\mathbb{C}}\{(A^\dagger)_1^{r_1}(A^\dagger)_2^{r_2} \cdots (A^\dagger)_{2n}^{r_{2n}} v_0 \mid r_1, \ldots, r_{2n} \in \mathbb{N}_0\}.$$

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The (infinitesimal) Stone - von Neumann representation

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This is a unitary representation.
A simple but key Lemma:

$$(A^\dagger)^2 A - 2(A^\dagger A A^\dagger) + A(A^\dagger)^2 = 0$$
$$A^2 A^\dagger - 2(A A^\dagger A) + A^\dagger A^2 = 0$$

Constructing $su(n, n)$

Set

$$\begin{array}{rclcrcl} e & = & A_n^\dagger A_{n+1}^\dagger & ; & f & = & A_n A_{n+1} \\ H_k & = & A_k A_k^\dagger & ; & h_e & = & H_n + H_{n+1} - 1 \\ \mu_k^+ & = & A_k^\dagger A_{k+1} & ; & \mu_k^- & = & A_k A_{k+1}^\dagger \ (k < n) \\ \nu_k^+ & = & A_{k+1}^\dagger A_k & ; & \nu_k^- & = & A_{k+1} A_k^\dagger \ (k > n). \end{array}$$

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Minors

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$$X = \begin{pmatrix} A_1 \\ A_2 \\ \cdots \\ A_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} A_{n+1} \\ A_{n+2} \\ \cdots \\ A_{2n} \end{pmatrix}$$

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$$W_k = X_k \cdot Y_k^T$$

is of rank k, hence all $(k+1) \times (k+1)$ minors vanish.

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The **ladder series** of mass 0 and spin $\frac{i}{2}$, resp. $\frac{j}{2}$.

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$$\mathcal{U}_q(\mathfrak{g}^{\mathbb{C}}) = \mathcal{A}_q^- \cdot \mathcal{U}_q(\mathfrak{k}^{\mathbb{C}}) \cdot \mathcal{A}_q^+ .$$

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$$\begin{aligned}
Z_{ij}Z_{ik} &= q^{-1}Z_{ik}Z_{ij} \text{ if } j < k; \\
Z_{ij}Z_{kj} &= q^{-1}Z_{kj}Z_{ij} \text{ if } i < k; \\
Z_{ij}Z_{st} &= Z_{st}Z_{ij} \text{ if } i < s \text{ and } t < j; \\
Z_{ij}Z_{st} &= Z_{st}Z_{ij} - (q - q^{-1})Z_{it}Z_{sj} = \text{ if } i < s \text{ and } j < t.
\end{aligned}$$

Quantized modules

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Quantized modules

$M(V_{\tau'}) \to M_q(V_{\tau'})$ (straightforward)

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Duality "essentially" the q Killing form

q homomorphisms

$$M_q(V_{\tau_1'}) \xrightarrow{\phi_q} M_q(V_{\tau'}).$$

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$$(W_1, W_2, \cdots, W_{N_1}) \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{N1} \\ p_{12} & p_{22} & \cdots & p_{N2} \\ \vdots & \vdots & \cdots & \vdots \\ p_{1N_1} & p_{2N_1} & \cdots & L_{p_{NN_1}} \end{pmatrix} \in \mathcal{A}^- \otimes V_{\Lambda}.$$

q homomorphisms

We set $\underline{\underline{\phi}}_{\Lambda,\Lambda_1} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{N1} \\ p_{12} & p_{22} & \cdots & p_{N2} \\ \vdots & \vdots & \ldots & \vdots \\ p_{1N_1} & p_{2N_1} & \cdots & L_{p_{NN_1}} \end{pmatrix}$. Notice that if $\phi_{\Lambda,\Lambda_1} \circ \phi_{\Lambda_1,\Lambda_2} = \phi_{\Lambda,\Lambda_2}$ then $\underline{\underline{\phi}}_{\Lambda,\Lambda_2} = \underline{\underline{\phi}}_{\Lambda_1,\Lambda_2} \cdot \underline{\underline{\phi}}_{\Lambda,\Lambda_1}$.

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See also J: Special Classes of Homomorphisms between generalized Verma modules for $U_q(su(n,n))$ (10 pp) 2019 J. Phys.: Conf. Ser. 1194 012055

But what is a q differential operator actually?

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$$Z^{\mathbf{a}} := Z_{11}^{a_{11}} Z_{21}^{a_{21}} \cdots Z_{n1}^{a_{n1}} Z_{12}^{a_{12}} \cdots Z_{nn}^{a_{nn}}, \tag{2}$$

$$W^{\mathbf{a}} := W_{11}^{a_{11}} W_{21}^{a_{21}} \cdots W_{n1}^{a_{n1}} W_{12}^{a_{12}} \cdots W_{nn}^{a_{nn}}. \tag{3}$$

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$$\begin{aligned} [[a]]_q &= 1 + q^2 + \cdots + q^{2a-2} \ (J), \\ \{\{a\}\}_q &= 1 + q^{-2} + \cdots + q^{-(2a-2)} \ (K), \text{ and} \\ [a]_q &= q^{-a+1} + \cdots + q^{a-1} \ (L). \end{aligned}$$

$$\left(Z^{\mathbf{a}}, W^{\mathbf{b}}\right)_J = \left(\frac{-1}{q - q^{-1}}\right)^{|\mathbf{a}|} \delta_{\mathbf{a},\mathbf{b}} [[\mathbf{a}]]_q!$$

Is left right?

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$$\left(\widehat{\partial^R_{xy}} Z^{\mathbf{a}}, W^{\mathbf{b}}\right) \quad = \quad \left(Z^{\mathbf{a}}, W^{\mathbf{b}} W_{xy}\right)$$

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$$\begin{aligned}\left(\widehat{\partial^R_{xy}} Z^{\mathbf{a}}, W^{\mathbf{b}}\right) &= \left(Z^{\mathbf{a}}, W^{\mathbf{b}} W_{xy}\right) \\ \left(\widehat{\partial^L_{xy}} Z^{\mathbf{a}}, W^{\mathbf{b}}\right) &= \left(Z^{\mathbf{a}}, W_{xy} W^{\mathbf{b}}\right)\end{aligned}$$

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\left(\widehat{\partial^R_{xy}} Z^{\mathbf{a}}, W^{\mathbf{b}}\right) &= \left(Z^{\mathbf{a}}, W^{\mathbf{b}} W_{xy}\right) \\
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\left(Z_{xy} Z^{\mathbf{a}}, W^{\mathbf{b}}\right) &; \left(Z^{\mathbf{a}} Z_{xy}, W^{\mathbf{b}}\right)
\end{aligned}
$$

The quantized Weyl algebra

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\begin{aligned}
M_{ij}(Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}}) &= Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}+1} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}} \\
D_{ij}(Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}}) &= [a_{ij}]_q Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}-1} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}} \\
H_{ij}^{\pm 1}(Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}}) &= q^{\pm a_{ij}} Z_{11}^{a_{11}} \cdot\cdot Z_{ij}^{a_{ij}+1} \cdot\cdot\cdot\cdot Z_{nn}^{a_{nn}}
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\end{aligned}$$

$$\begin{aligned}
D_{ij}M_{ij} - qM_{ij}D_{ij} &= H_{ij}^{-1} \\
D_{ij}M_{ij}^{o} - q^{-1}M_{ij}D_{ij} &= H_{ij} \\
H_{ij}D_{ij} &= q^{-1}D_{ij}H_{ij} \\
H_{ij}M_{ij} &= qM_{ij}H_{ij}.
\end{aligned}$$

A Result

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Proposition[arXiv:1905.04478v1 [math.QA]] In the pairing indexed by L, the algebra of covariant differential operators is equal to the quantum Weyl algebra

Where do the (unitary!) (q-) covariant differential operators come from?

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Proposition. In $\mathcal{U}_q(su(N,N))$ there is a (unitary) scalar highest weight module Π_k in which all $k \times k$ minors vanish.

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Consider

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Consider

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Both vectors $W_{1,n+1}^{R-1}W_{2,n+1}$ and $W_{1,n+1}^R$ are in the h.w. module generated by $W_{1,n+1}^R$, but the vector vanishes, i.e. there is a homomorphism!!! into $M_q(W_{1,n+1}^R)$.

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Definition The support of $M(V_\Lambda)$, $\Lambda = (\Lambda_0, \lambda)$, is the smallest $r > 0$ for which there exists a $M(V_{\Lambda_1})$ and a homomorphism $\psi : M(V_{\Lambda_1}) \to M(V_\Lambda)$ such that

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Theorem We get all unitary representations of support $r = 1, \ldots, n$ this way.

The Hayashi-Weyl Algebra

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T. Hayashi, q-analogues of Clifford and Weyl Algebras–Spinor and Oscillator Representations of Quantum Enveloping Algebras, Comm. Math. Phys. **127**, Number 1 (1990), 129-144.

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We start with a quantum analog for the Weyl algebra $\mathcal{A}_q = \mathcal{A}_q^-(N)$ introduced by Hayashi. Let q be a non-zero complex number such that $q^4 \neq 1$. Then the algeba is defined as an associative unital algebra with generators $\psi_i, \psi_i^*, \omega_i^{\pm 1}$ and relations

$$\begin{aligned}
&\omega_i\omega_j = \omega_j\omega_i, \quad \omega_i\omega_i^{-1} = \omega_i^{-1}\omega_i = 1,\\
&\psi_i\psi_j = \psi_j\psi_i, \quad \psi_i^*\psi_j^* = \psi_j^*\psi_i^*,\\
&\psi_i\psi_j = \psi_j\psi_i \quad i \neq j,\\
&\omega_i\psi_j\omega_i^{-1} = q^{-\delta_{ij}}\psi_j, \quad \omega_i\psi_j^*\omega_i^{-1} = q^{\delta_{ij}}\psi_j^*,\\
&\psi_i\psi_i^* - q^2\psi_i^*\psi_i = \omega_i^{-2}, \quad \psi_i\psi_i^* - q^{-2}\psi_i^*\psi_i = \omega_i^2.
\end{aligned}$$

The last pair of relations is equivalent to the following

$$\psi_i\psi_i^* = \frac{(q\omega_i)^2 - (q\omega_i)^{-2}}{q^2 - q^{-2}}, \quad \psi_i^*\psi_i = \frac{\omega_i^2 - \omega_i^{-2}}{q^2 - q^{-2}}.$$

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This is just the quantized Weyl algebra. The introduced algebra has a natural representation in infinite-dimensional vector space $V^- = \{v(\mathbf{m}) | \mathbf{m} \in \mathbb{Z}_+^{\mathbf{N}}\}$. Namely,

$$\omega_i v(\mathbf{m}) = q^{m_i} v(\mathbf{m}), \quad \psi_i v(\mathbf{m}) = [m_i]_{q^2} v(\mathbf{m}-\mathbf{e}_i), \quad \psi_i^* v(\mathbf{m}) = v(\mathbf{m}+\mathbf{e}_i),$$

where $\mathbf{m} = \{m_1, \ldots, m_N\}$ and $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$.
Let us consider $N = 2n$ and obtain explicit formulas for embedding of $U_q\mathfrak{sl}_{2n}$ into $\mathcal{A}_q$. Formulas for simple roots remain the same

$$\begin{aligned} E_n &= i\psi_n\psi_{n+1}, & F_n &= i\psi_n^*\psi_{n+1}^*, \\ E_j &= \psi_j\psi_{j+1}^*, & F_j &= \psi_j^*\psi_{j+1}, \quad j < n, \\ E_j &= \psi_j^*\psi_{j+1}, & F_j &= \psi_j\psi_{j+1}^*, \quad j > n, \end{aligned}$$

The Hayashi-Weyl algebra and $\mathcal{U}_q(su(n,n))$

$$K_n = q^{-2}\omega_n^{-2}\omega_{n+1}^{-2}; \qquad E_nF_n - F_nE_n = \frac{K_n - K_n^{-1}}{q^2 - q^{-2}}$$

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$$\begin{pmatrix} (-q)^{-2(n-1)}\prod_2^n \omega_i^{-2}\psi_1\psi_{n+1} & (-q)^{-2n}\prod_2^{n+1}\omega_i^{-2}\psi_1\psi_{n+2} & \cdots \\ (-q)^{-2(n-2)}\prod_3^n \omega_i^{-2}\psi_2\psi_{n+1} & (-q)^{-2(n-1)}\prod_3^{n+1}\omega_i^{-2}\psi_2\psi_{n+2} & \cdots \\ \vdots & \vdots & \\ \psi_n\psi_{n+1} & (-q)^{-2}\omega_{n+1}^{-2}\psi_n\psi_{n+2} & \cdots \end{pmatrix} \tag{4}$$

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$$Z_{ij} = (i\psi_{n+1-i}\psi_{n+j})(-q^{-2})^{i+j-2} \prod_{x=1}^{i-1} \prod_{y=1}^{j-1} \omega_{n+1-x}^{-2} \omega_{n+j}^{-2}.$$

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D. Shklyarov, S. Sinel'shchikov, A. Stolin, and L. Vaksman, ON A q-ANALOGUE OF THE PENROSE TRANSFORM. In: LECTURES ON q-ANALOGUES OF CARTAN DOMAINS AND ASSOCIATED HARISH-CHANDRA MODULES L. Vaksman (Ed.) arXiv:math/0109198

$\mathcal{U}_q(su(n,n))$ and higher order minors

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S. Launois, T. H. Lenagan, and L. Rigal, Quantum Unique Factorisation Domains, J. London Math. Soc. **74**, 321-340 (2006)

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Thank you