On compact quantum groups of Lie type

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Compact quantum groups

Definition (Woronowicz/Dijkhuizen–Koornwinder)

A **compact quantum group** is a Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$ such that all its finite dimensional corepresentations are unitarizable.

Definition

We say a that a compact quantum group G is of **Lie type** if there is a (necessarily unique) compact connected Lie group H and a dimension-preserving isomorphism of representation semirings $R^+(G) \cong R^+(H)$. We then also say that G is of H-**type**.

Can we classify all compact quantum groups of Lie type?

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A few known complete results

$H = \mathrm{SU}(2)$:

- Woronowicz, Podleś–Müller: $\mathrm{SU}_q(2)$ ($q \in [-1,1], q \neq 0$);
- Banica, no restriction on dimension function: free orthogonal quantum groups (Van Daele–Wang);
- Bichon, no $*$-structure, no restriction on dimension: quantum groups of bilinear forms (Dubois-Violette–Launer).

$H = \mathrm{SO}(3)$:

- Mrozinski, no restriction on dimension: quantum automorphism groups of finite dimensional C*-algebras.

$H = \mathrm{SU}(3)$:

- Ohn, no $*$-structure: a long list of deformations, a number of which are multiparametric.

Non-Kac compact quantum groups of SU(n)-type

Theorem (Yamashita–N)

The following is a complete list of compact quantum groups of SU(n)*-type* ($n \geq 2$) *such that* $S^2 \neq \mathrm{id}$:

- *the* q*-deformations* $\mathrm{SU}_q(n)$ *for* $0 < q < 1$;
- *the twists* $\mathrm{SU}_q^\omega(n)$ *of* $\mathrm{SU}_q(n)$ *by 2-cocycles* $\omega \in Z^2(\hat{T}; \mathbb{T})$, *where* $T \subset \mathrm{SU}(n)$ *is the maximal torus (Artin–Schelter–Tate, Levendorskiy–Soibelman);*
- *the twists* $\mathrm{SU}_q^{\tau,\omega}(n)$ *of* $\mathrm{SU}_q^\omega(n)$ *by 3-cocycles* $\tau \in Z^3(Z(\widehat{\mathrm{SU}(n)}); \mathbb{T})$.

There are no "non-obvious" isomorphisms in this list.

Classification in terms of generators and relations

Assume $n \geq 2$. Take a number $0 < q \leq 1$, a matrix $\omega = (\omega_{ij})_{i,j=1}^n$ with entries of modulus one such that $\omega_{ii} = 1$, $\omega_{ji} = \bar{\omega}_{ij}$ and $\prod_i \omega_{ij} = 1$ for any j, and an $(n-1)$-tuple $\tau = (\tau_1, \dots, \tau_{n-1})$ of roots of unity of order n. Consider the algebra $\mathbb{C}[\mathrm{SU}_q^{\tau,\omega}(n)]$ with generators v_{ij}, $1 \leq i,j \leq n$, and relations

$$v_{ij} v_{il} = \Big(\prod_{j \leq p < l} \tau_p^{-1} \Big) q \bar{\omega}_{jl}^2 v_{il} v_{ij} \ (j < l), \quad v_{ij} v_{kj} = \Big(\prod_{i \leq p < k} \tau_p \Big) q \omega_{ik}^2 v_{kj} v_{ij} \ (i < k),$$

$$v_{ij} v_{kl} = \Big(\prod_{k \leq p < i} \tau_p^{-1} \Big) \Big(\prod_{j \leq p < l} \tau_p^{-1} \Big) \omega_{ik}^2 \bar{\omega}_{jl}^2 v_{kl} v_{ij} \ (i > k, j < l),$$

$$\Big(\prod_{j \leq p < l} \tau_p \Big) \omega_{jl}^2 v_{ij} v_{kl} - \Big(\prod_{i \leq p < k} \tau_p \Big) \bar{\omega}_{ki}^2 v_{kl} v_{ij} = (q - q^{-1}) v_{il} v_{kj} \ (i < k, j < l),$$

$$\sum_{\sigma \in S_n} \tau^{m(\sigma)} (-q)^{|\sigma|} \bar{\omega}(1, \dots, n) \omega(\sigma(1), \dots, \sigma(n)) v_{1\sigma(1)} \cdots v_{n\sigma(n)} = 1,$$

where $m(\sigma) = (m(\sigma)_1, \dots, m(\sigma)_{n-1})$ is the multi-index given by $m(\sigma)_i = \sum_{k=2}^{n} (k-1) m_i^{(k,\sigma(k))}$, with $m_i^{(k,j)} = 1$ if $k \le i < j$, $m_i^{(k,j)} = -1$ if $j \le i < k$ and $m_i^{(k,j)} = 0$ in the remaining cases, and where $\omega(i_1, \dots, i_n) = \prod_{k<l} \omega_{i_k, i_l}$.

The $*$-structure is uniquely determined by requiring the invertible matrix $(v_{ij})_{i,j}$ to be unitary.

Then $\mathbb{C}[\mathrm{SU}_q^{\tau,\omega}(n)]$ with the coproduct

$$\Delta(v_{ik}) = \sum_{j=1}^{n} v_{ij} \otimes v_{jk}$$

is a compact quantum group $\mathrm{SU}_q^{\tau,\omega}(n)$ of $\mathrm{SU}(n)$-type.

Theorem (Yamashita–N)

Every compact quantum group of $\mathrm{SU}(n)$*-type* $(n \geq 2)$ *such that* $S^2 \neq \mathrm{id}$ *is isomorphic to* $\mathrm{SU}_q^{\tau,\omega}(n)$ *for some* $q \in (0,1)$*,* ω *and* τ *as above.*

Furthermore, two such quantum groups $\mathrm{SU}_q^{\tau,\omega}(n)$ *and* $\mathrm{SU}_{q'}^{\tau',\omega'}(n)$ *are isomorphic if and only if* $q = q'$, $\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \tau_i'^i$ *and one of the following holds:*

(i) $\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \omega'^2_{ij} \prod_{k=i}^{j-1} \tau'_k$ *for all* $1 \leq i < j \leq n-1$*;*

(ii) $\omega_{ij}^2 \prod_{k=i}^{j-1} \tau_k = \omega'^2_{n-i+1,n-j+1} \prod_{k=i}^{j-1} \bar{\tau}'_{n-k}$ *for all* $1 \leq i < j \leq n-1$*.*

Categorical reformulation of the classification problem

Woronowicz's Tannaka–Krein duality says that a compact quantum group is a **rigid $\mathbf{C}^*$-tensor category** $\mathcal{C}$ plus a **unitary fiber functor** $\mathcal{C} \to \mathrm{Hilb}_f$.

Hence classification of compact quantum groups of H-type can be divided into the following three problems:

- classification of rigid C^*-tensor categories $\mathcal{C}$ with fusion rules of H;
- classification of monoidal autoequivalences of $\mathcal{C}$;
- classification of unitary fiber functors $\mathcal{C} \to \mathrm{Hilb}_f$ defining the same dimension function on the objects as the usual dimension of representations of H.

Tensor categories of Lie type

Theorem (Kazhdan–Wenzl / Jordans)

Any rigid C^*-tensor category $\mathcal{C}$ with fusion rules of $\mathrm{SU}(n)$ *is unitarily monoidally equivalent to the twisting of the category* $\mathrm{Rep}\,\mathrm{SU}_q(n)$*,* $0 < q \leq 1$*, by a* $\mathbb{T}$*-valued 3-cocycle* τ *on the dual of the center of* $\mathrm{SU}(n)$*. The number q and the cohomology class*

$$[\tau] \in H^3(Z(\widehat{\mathrm{SU}(n)}); \mathbb{T}) \cong \mathbb{Z}/n\mathbb{Z}$$

are uniquely determined by $\mathcal{C}$*.*

To simplify matters from now on we will only consider the categories $\mathrm{Rep}\,\mathrm{SU}_q(n)$ and, more generally, $\mathrm{Rep}\,G_q$ for compact connected Lie groups G.

Autoequivalences of representation categories

Theorem (Tuset–N)

For any compact connected group G, we have a canonical isomorphism

$$\mathrm{Aut}^{\otimes}(\mathrm{Rep}\, G) \cong H^2(\widehat{Z(G)}; \mathbb{T}) \rtimes \mathrm{Out}(G).$$

If G is a compact connected semisimple Lie group, then we also have

$$\mathrm{Aut}^{\otimes}(\mathrm{Rep}\, G_q) \cong \mathrm{Aut}^{\otimes}(\mathrm{Rep}\, G)$$

for any $0 < q < 1$.

Rigid tensor categories

Briefly, a rigid C*-tensor category is a category $\mathcal{C}$ such that:

- $\mathrm{End}_{\mathcal{C}}(U)$ is a finite dimensional C*-algebra for every object U;
- tensor products $U \otimes V$ are defined and we are given natural unitary isomorphisms
$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W);$$
- there is a **unit object** $\mathbb{1}$ such that
$$U \otimes \mathbb{1} \cong \mathbb{1} \otimes U \cong U;$$
- for every object U there is a **dual object** $\bar{U}$: there exist morphisms $R_U \colon \mathbb{1} \to \bar{U} \otimes U$ and $\bar{R}_U \colon \mathbb{1} \to U \otimes \bar{U}$ such that
$$(R_U^* \otimes \iota)(\iota \otimes \bar{R}_U) = \iota_{\bar{U}}, \quad (\bar{R}_U^* \otimes \iota)(\iota \otimes R_U) = \iota_U.$$

Dimension function

Every rigid C*-tensor category has a canonical spherical structure and therefore a notion of dimension of objects. Explicitly,

$$d^{\mathcal{C}}(U) = \min_{(R_U, \bar{R}_U)} \|R_U\| \, \|\bar{R}_U\|.$$

Denote by $I_{\mathcal{C}}$ the set of isomorphism classes of simple object in $\mathcal{C}$. For every $s \in I_{\mathcal{C}}$ fix a representative U_s.

For every object U let $\Gamma_U = (m^U_{st})_{s,t \in I_{\mathcal{C}}}$, where m^U_{st} is the multiplicity of U_s in $U \otimes U_t$. For any dimension function d on $\mathcal{C}$ we have

$$\|\Gamma_U\| \leq d(U).$$

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Amenability

Definition (Hiai–Izumi)

- A dimension function d on $\mathcal{C}$ is called **amenable** if $\|\Gamma_U\| = d(U)$ for all objects U. If $d = d^{\mathcal{C}}$, then $\mathcal{C}$ itself is called amenable.
- A dimension function d on $\mathcal{C}$ is called **weakly amenable** if there is a state on $\ell^\infty(I_{\mathcal{C}})$ invariant under the operators $d(U)^{-1}d^{-1}\Gamma_U d$ for all U. If $d = d^{\mathcal{C}}$, then $\mathcal{C}$ itself is called weakly amenable.

Theorem (ess. Banica)

Assume H is a compact group and G is a compact quantum group of H-type. Then

- *the classical dimension function* $\dim$ *on* $\operatorname{Rep} G$ *is amenable;*
- *the quantum dimension function* $\dim_q = d^{\operatorname{Rep} G}$ *on* $\operatorname{Rep} G$ *is weakly amenable.*

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Universal amenable functor

Theorem (Yamashita–N)

Assume $\mathcal{C}$ is a weakly amenable rigid C^*-tensor category. Then there exists a universal unitary tensor functor $\Pi\colon \mathcal{C} \to \mathcal{P}$ into a rigid C^*-tensor category $\mathcal{P}$ such that

$$\|\Gamma_U\| = d^{\mathcal{P}}(\Pi(U)) \quad \text{for all objects} \quad U.$$

Furthermore, the category $\mathcal{P}$ is amenable; in particular, we can take $\mathcal{P} = \mathcal{C}$ and $\Pi = \mathrm{id}$ if and only if $\mathcal{C}$ is amenable.

The pair $(\mathcal{P}, \Pi)$ is called the **Poisson boundary** of $\mathcal{C}$.

Construction of the Poisson boundary

Let $\mathcal{C}$ be a rigid C*-tensor category. For every object U consider the functor F_U of tensoring by U on the right.

Denote by $\hat{\mathcal{C}}$ the category with the same objects as $\mathcal{C}$ but with morphisms

$$\hat{\mathcal{C}}(U,V) = \mathrm{Nat}_{\mathrm{b}}(F_U, F_V) \cong \ell^\infty\text{-}\bigoplus_{s\in I_{\mathcal{C}}} \mathcal{C}(U_s\otimes U, U_s\otimes V).$$

In particular, $\mathrm{End}_{\hat{\mathcal{C}}}(\mathbb{1}) \cong \ell^\infty(I_{\mathcal{C}})$.

This is a C*-tensor category, with the tensor product of morphisms $\eta\colon U\to V$ and $\omega\colon W\to Z$ defined by

$$(\eta\otimes\omega)_X = (\eta_X\otimes\iota_Z)\omega_{X\otimes U}.$$

Harmonic elements

For every $s \in I_{\mathcal{C}}$ fix morphisms $R_s, \bar{R}_s$ defining the dual $\bar{U}_s$ such that $\|R_s\| = \|\bar{R}_s\| = d^{\mathcal{C}}(U_s)^{1/2}$.
Define a partial trace operator P_s on $\hat{\mathcal{C}}(U, V)$ by

$$P_s(\eta)_X = d^{\mathcal{C}}(U_s)^{-1}(R_s^* \otimes \iota_X \otimes \iota_V)(\iota_{\bar{U}_s} \otimes \eta_{U_s \otimes X})(R_s \otimes \iota_X \otimes \iota_U).$$

Definition

We say that an element $\eta \in \hat{\mathcal{C}}(U, V)$ is **absolutely harmonic** if $P_s(\eta) = \eta$ for all $s \in I_{\mathcal{C}}$.

We define $\mathcal{P}$ as the category with the same objects as $\mathcal{C}$ and morphisms the absolutely harmonic morphisms in $\hat{\mathcal{C}}$.
(More precisely, we take the idempotent completion of this category.)

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Product at infinity

Assuming that $I_{\mathcal{C}}$ is at most countable, it can be shown that weak amenability of $\mathcal{C}$ is equivalent to existence of a probability measure μ on $I_{\mathcal{C}}$ such that the constants are the only functions in $\ell^\infty(I_{\mathcal{C}})$ invariant under the operator

$$P_\mu = \sum_{s\in I_{\mathcal{C}}} \mu(s) P_s.$$

Then the composition of morphisms in $\mathcal{P}$ is defined by

$$(\eta\cdot\omega)_X = \lim_{n\to\infty} P_\mu^n(\eta\omega)_X.$$

(It can be shown that this gives the unique product such that $\mathcal{P}$ becomes a C*-tensor category with the same notion of positivity of morphisms as in $\hat{\mathcal{C}}$.)

Maximal quantum subgroups of Kac type

The **maximal quantum subgroup of Kac type** of a compact quantum group G is defined by $\mathbb{C}[K] = \mathbb{C}[G]/\langle x - S^2(x)\rangle$. (Soltan)

Theorem (Yamashita–N)

Assume G is a compact quantum group of H-type for a compact group H. Let K be its maximal quantum subgroup of Kac type. Then the Poisson boundary of $\mathrm{Rep}\, G$ *is the forgetful functor* $\mathrm{Rep}\, G \to \mathrm{Rep}\, K$.

It is not difficult to show that the maximal quantum subgroup of Kac type of G_q, $0 < q < 1$, for a compact connected Lie group G is the maximal torus $T \subset G$. (Tomatsu)

Corollary (Yamashita–N)

For $0 < q < 1$, any dimension-preserving unitary fiber functor $\mathrm{Rep}\, G_q \to \mathrm{Hilb}_f$ *factors in a unique way through* $\mathrm{Rep}\, T$. *Therefore the isomorphism classes of such functors are parametrized by $H^2(\hat{T}; \mathbb{T})$.*

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