Representation theory for Lie algebras of vector fields

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June 5, 2019

Part I

Background

The classical Lie algebras

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra.

Then $\mathfrak{g}$ contains a self-normalizing nilpotent subalgebra $\mathfrak{h}$, called a **Cartan subalgebra**.

For $\alpha \in \mathfrak{h}^*$ we have a corresponding **root-space**

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

This yields the **root-space decomposition** $\mathfrak{g} = \bigoplus_{\alpha \in h^*} \mathfrak{g}_\alpha$.

The $\alpha \in h^*$ corresponding to nonzero rootspaces $\Phi := \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0\}$ is called the **root system** of $\mathfrak{g}$.

Classification

Classification of simple finite dimensional Lie algebras (Cartan 1894)

Simple finite-dimensional complex Lie algebras are determined up to isomorphism by their root systems. These consist of four infinite families and five exceptional cases

$$A_n,\ B_n,\ C_n,\ D_n, \qquad E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

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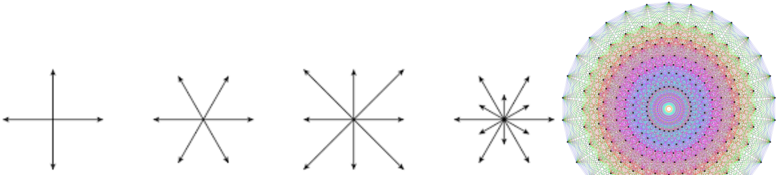

$A_1 \times A_1$ A_2 $B_2 \cong C_2$ G_2

Representation theory

The most fundamental modules for such Lie algebras $\mathfrak{g}$ are the **weight modules**:

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \text{ where } M_\lambda = \{m \in M \mid h \cdot m = \lambda(h)m \text{ for all } h \in \mathfrak{h}^*\}$$

Every finite dimensional module is a weight module.

Historically several other categories of modules were studied, for example parabolically induced modules, Whittaker modules, Gelfand-Zetlin modules, $U(\mathfrak{h})$-free modules.

Part II

Lie algebras of vector fields

Lie algebras of vector fields, geometrically

Let k be a field with $k = \overline{k}$, *char* $k = 0$.

Let $X \subset \mathbb{A}^n$ be an irreducible affine algebraic variety over k.

Let $\mathcal{V}_X$ be the space of vector fields on X, geometrically $\mathcal{V}_X = \Gamma(TX)$.

The vector fields $\mathcal{V}_X$ form a Lie algebra under the Lie derivative:

$$[x, y] := \mathfrak{L}_x \cdot y$$

Lie algebras of vector fields, algebraically

Let $X \subset \mathbb{A}^n$ be an irreducible affine algebraic variety over k.

Let I_X be the ideal of polynomials vanishing on X.

$A_X := k[x_1, \dots, x_n]/I_X$ is the algebra of polynomial functions on X.

Then $\mathcal{V}_X = \mathrm{Der}_k(A_x)$ is the Lie algebra of polynomial vector fields on X. The bracket is given by

$$[x, y] := x \circ y - y \circ x$$

Note that A_X is a $\mathcal{V}_X$-module and that $\mathcal{V}_X$ is an A_X-module.

Realizations

Example

For $X = \mathbb{A}^n$, we have
$A = k[x_1, \ldots, x_n]$ and
$\mathfrak{L} := \mathcal{V}_X = \bigoplus_{i=1}^n k[x_1, \ldots, x_n]\frac{\partial}{\partial x_i}$

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$\mathfrak{L} := \mathcal{V}_X = \bigoplus_{i=1}^n k[x_1, \dots, x_n]\frac{\partial}{\partial x_i}$

For general X we can view $\mathcal{V}_X$ as a subquotient of $\mathfrak{L}$:

$$\mathcal{V}_X \simeq \frac{\{\mu \in \mathfrak{L} \mid \mu(I_X) \subset I_X\}}{\{\mu \in \mathfrak{L} \mid \mu(k[x_1, \dots, x_n]) \subset I_X\}}$$

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Another way is to realize $\mathcal{V}$ as a subalgebra of $\bigoplus_{i=1}^n A\frac{\partial}{\partial x_i}$ as follows:
Let $I = \langle g_1, \dots, g_m \rangle$ and take $J = (\frac{\partial g_i}{\partial x_j})$.
Then $\sum f_i \frac{\partial}{\partial x_i} \in \mathcal{V}$ if and only if $(f_1, \dots, f_n) \in Ker\ J$.

Example: The circle

Example

For $X = \mathbb{S}^1$ we have

$$A = k[x,y]/\langle x^2 + y^2 - 1 \rangle = k[x] \oplus yk[x]$$

$J = (2x \quad 2y)$ so for $f_1, f_2 \in A$ we have $f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} \in \mathcal{V}$ if and only if $xf_1 + yf_2 = 0$ in A. Thus $\mathcal{V} = A(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$.

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Now assume that $k = \mathbb{C}$, take $t = x + iy$, $s = x - iy$. Then $s = t^{-1}$, so we have $A = k[t, t^{-1}]$ and $\mathcal{V} = k[t, t^{-1}]\frac{\partial}{\partial t}$, which is called the **Witt algebra**.

Example: The 2-sphere

Example

For $X = \mathbb{S}^2$ we have

$$A = k[x, y, z]/\langle x^2 + y^2 + z^2 - 1\rangle$$

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Let

$$\Delta_{12} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \Delta_{23} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad \Delta_{31} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$

Then $\mathcal{V}$ is generated over A by $\{\Delta_{12}, \Delta_{23}, \Delta_{31}\}$, and

$$z\Delta_{12} + x\Delta_{23} + y\Delta_{31} = 0$$

Properties of $\mathcal{V}_X$

Proposition

- $\mathcal{V}_X$ is simple $\Leftrightarrow$ X is smooth (D. Jordan 2000).
- $\mathcal{V}_X$ does not admit a Cartan subalgebra in general.
- $\mathcal{V}_X$ may not contain nonzero semisimple or nilpotent elements.

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Thus the classical approach to representation theory fails. However, when X is a torus, it has been proved that every $\mathcal{V}_X$ module M has a natural cover $\widehat{M} \to M$ where $\widehat{M}$ is a module over both $\mathcal{V}_X$ and A_X.

Part III

Representation theory

Papers

Based on joint work with Yuly Billig, Vyasheslav Futorny, André Zaidan

- "Representations of the Lie algebra of vector fields on a sphere" (joint with Y. Billig). Journal of pure and applied algebra (223) 3581–3593.
- "Representations of Lie algebras of vector fields on affine varieties" (joint with Y. Billig and V. Futorny). Accepted for publication in Israel Journal of Mathematics. Preprint: arXiv:1709.08863.
- "Gauge modules for the Lie algebra of vector fields on affine varieties." (joint with Y. Billig and A. Zaidan). Submitted. Preprint: arXiv:1903.02626.

$A\mathcal{V}$-modules

Definition

The category of $A\mathcal{V}$-modules has as objects spaces M equipped with A- and $\mathcal{V}$-module structures that are compatible in the following sense:

$$\eta \cdot f \cdot m = \eta(f) \cdot m + f \cdot \eta \cdot m$$

for all $\eta \in \mathcal{V}$, $f \in A$, and $m \in M$.
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Proposition

$$A\mathcal{V}\text{-Mod} \simeq A\#U(\mathcal{V})\text{-Mod}$$

Categorical properties

Let $M, N \in A\mathcal{V}$-Mod.

Definition

We define $M \otimes N := M \otimes_A N$ with the following A and $\mathcal{V}$-actions

$$f \cdot (m \otimes n) := (f \cdot m) \otimes n \text{ and } \quad \eta \cdot (m \otimes n) = (\eta \cdot m) \otimes n + m \otimes (\eta \cdot n)$$

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$$(f \cdot \varphi)(m) := \varphi(f \cdot m) \quad \text{and} \quad (\eta \cdot \varphi)(m) := \eta(\varphi(m)) - \varphi(\eta \cdot m)$$

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Then $M \otimes N$ and M° are in $A\mathcal{V}$-Mod too.
So $A\mathcal{V}$ is a monoidal abelian catgory equipped with a dual.

Shen's modules

Example

Let $X = \mathbb{A}^n$, let U be a $\mathfrak{gl}_n$-module, and let $b = (b_1, \dots, b_n)$ with $b_i \in k[x_1, \dots, x_n]$ such that $\frac{\partial b_i}{\partial x_j} = \frac{\partial b_j}{\partial x_i}$.

Then the space $k[x_1, \dots, x_n] \otimes U$ is a natural A-module and it becomes an $A\mathcal{V}$-module if we define the following $\mathcal{V}$-action:

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \cdot (g \otimes u) := \sum_{i=1}^{n} (f_i \frac{\partial g}{\partial x_i} + f_i g b_i) \otimes u + g \sum_{i,p=1}^{n} \frac{\partial f_i}{\partial x_p} \otimes E_{p,i} \cdot u$$

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These modules were studied by G. Shen (1986). They appear from choosing certain natural embeddings

$$\mathcal{V}_X \subset \mathrm{Cur}(X) := \mathcal{V}_X \ltimes (\mathfrak{gl}_n(A) \oplus A).$$

An atlas for X

Let $X \subset \mathbb{A}^n$ be a smooth irreducible affine variety. For $h \in A$, we define the corresponding chart as

$$N(h) := \{P \in X \mid h(P) \neq 0\}$$

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Let $J = (\frac{\partial g_i}{\partial x_j})$, let $r = \mathrm{rank}_{Frac(A)} J$, and let Φ be the set of nonzero $r \times r$ minors in J. Then

$$X = \bigcup_{h \in \Phi} N(h).$$

We call this the standard atlas for X.

Chart parameters

Definition

We say that $t_1, \ldots, t_s \in A$ are **chart parameters** in the chart $N(h)$ provided that the following conditions are satisfied:

- $t_1, \ldots, t_s$ are algebraically independent, so that $k[t_1, \ldots, t_s] \subset A$.
- Each element of A is algebraic over $k[t_1, \ldots, t_s]$.
- The derivation $\frac{\partial}{\partial t_i}$ of $k[t_1, \ldots, t_s]$ extends uniquely to a derivation of the localized algebra $A_{(h)}$.

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Remark

- $s = \dim X$
- We may always pick $\{t_1, \ldots, t_s\} \subset \{x_1, \ldots, x_n\}$.
- $\mathrm{Der}(A_{(h)}) = \bigoplus_{i=1}^{s} A_{(h)} \frac{\partial}{\partial t_i}$

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Thus each $\eta \in \mathrm{Der}(A)$ has a unique representation $\eta = \sum_{i=1}^{s} f_i \frac{\partial}{\partial t_i}$.

Example: the 2-sphere

Example

Let $X = \mathbb{S}^2$. Then $A = k[x, y, z]/\langle x^2 + y^2 + z^2 - 1\rangle$. We have $J = (2x\ 2y\ 2z)$ so our standard atlas is

$$X = N(x) \cup N(y) \cup N(z).$$

Now x, y are chart parameters in $N(z)$: the derivation $\frac{\partial}{\partial x}$ of $k[x, y]$ extends uniquely to a derivation of $A_{(z)}$ by defining $\frac{\partial}{\partial x}(z) = -\frac{x}{z}$. (This is necessary in order that $\frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 1) = 0 = \frac{\partial}{\partial x}(0)$)

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Thus we have an embedding of $\mathcal{V}$ into $\mathrm{Der}(A_{(z)})$, where the image is generated over A by three elements:

$$\mathrm{Der}(A) = A(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + Az\frac{\partial}{\partial x} + Az\frac{\partial}{\partial y} \subset \mathrm{Der}(A_{(z)}).$$

Generalization of Shen's modules

Let $X \subset \mathbb{A}^n$, fix a chart $N(h)$ and chart parameters $t_1, \ldots, t_s$. Let U be a simple $\mathfrak{gl}_s$-module, and let $b = (b_1, \ldots, b_s)$ with $b_i \in A_{(h)}$ such that $\frac{\partial b_i}{\partial x_j} = \frac{\partial b_j}{\partial x_i}$.

Then the space $A_{(h)} \otimes U$ becomes an $A\mathcal{V}$-module as follows:

$$\sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \cdot (g \otimes u) := \sum_{i=1}^{n} (f_i \frac{\partial g}{\partial t_i} + f_i g b_i) \otimes u + g \sum_{i,p=1}^{n} \frac{\partial f_i}{\partial t_p} \otimes E_{p,i} \cdot u$$

Note that these modules are not finitely generated over A.

Gauge modules

We have a doubly infinite filtration of $A_{(h)} \otimes U$:

$$\cdots \subset h^{k+1}A \otimes U \subset h^k A \otimes U \subset h^{k-1}A \otimes U \subset \cdots$$

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Let M be an $A\mathcal{V}$-submodule of $A_{(h)} \otimes U$.
We say that M is **bounded** if $M \subset h^k A \otimes U$ for some $k \in \mathbb{Z}$.

Definition

A bounded submodule of $A_{(h)} \otimes U$ is called a **local gauge module**.
An $A\mathcal{V}$-module which is isomorphic to a local gauge module in each standard chart $N(h)$ is called a **gauge module**.

Examples

Examples

$A_X, \mathcal{V}_X$, and Ω^1_X are all gauge modules. In a fixed chart $N(h)$ they embed as follows:

$$A \simeq A \otimes k \subset A_{(h)} \otimes k \text{ where } k \text{ is the trivial } \mathfrak{gl}_s\text{-module}$$

$$\Omega^1_X \subset A_{(h)} \otimes N \text{ where } N \text{ is the natural } \mathfrak{gl}_s\text{-module}$$

$$\mathcal{V}_X \subset A_{(h)} \otimes N^* \text{ where } N^* \text{ is the co-natural } \mathfrak{gl}_s\text{-module}$$

Rank 1 example for the sphere

Example

Let U be one-dimensional, spanned by u_α where $I \cdot u_\alpha = \alpha u_\alpha$ for some $\alpha \in k$.

In this case we can show that the $A\mathcal{V}$-module $A_{(z)} \otimes u_\alpha$ contains a bounded submodule if and only if $\alpha \in 2\mathbb{Z}$. This local gauge module has form $z^{-\alpha} A \otimes u_\alpha$.

Simplicity

Theorem

Let M be a gauge module corresponding to a simple $\mathfrak{gl}_s$-module U. Then M is simple as an $A\mathcal{V}$-module.

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Proof sketch:

- Let $M' \subset M$ be a submodule
- Then $I = \{f \in A \mid f \cdot M \subset M'\}$ is a (chart independent) ideal of A.
- Submodules are closed under the operators $h \otimes E_{ij}$.
- For each standard chart $N(h)$, we have $h^k \in I$ for some k.
- Hilbert's weak nullstellensatz gives $1 \in I$ so $M = M'$.

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Conjecture

Every A-finitely generated simple $A\mathcal{V}$-module is a gauge module.

Restricting to $\mathcal{V}$

Simplicity of gauge modules in $\mathcal{V}$-Mod

Let M be a gauge-module corresponding to a simple $\mathfrak{gl}_s$-module U. Then M is simple as a $\mathcal{V}$-module if $U \not\simeq \Lambda^k N$ where N is the natural $\mathfrak{gl}_s$-module.

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When $U = \Lambda^k N$, reducible gauge modules appear in the de Rham complex:

$$A \to \Omega_X^1 \to \Omega_X^2 \to \cdots \to \Omega_X^{s-1}$$

here each Ω_X^k can be embedded in $A_{(h)} \otimes \Lambda^k N$ for each chart $N(h)$. Submodules are given by kernels/images of the differential.

Part IV

Quantum generalization

Quantum torus

Start with a complex matrix $q = (q_{ij})_{d \times d}$ with $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$. Let A_q be the unital associative algebra generated by t_i and t_i^{-1} subject to the relations

$$t_i t_j = q_{ij} t_j t_i \text{ and } t_i t_i^{-1} = 1.$$

This algebra is called the **quantum torus** associated to q.
Let $\mathcal{V}_q = \mathrm{Der}(A_q)$ be the Lie algebra of derivations of A_q.
We can now similarly define the category of $A_q\mathcal{V}_q$-modules as spaces M with compatible A_q and $\mathcal{V}_q$ module structures.

Differences in the quantum setting

Both A_q and $\mathcal{V}_q$ are $\mathbb{Z}^d$-graded.
For $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ we write t^n for $t_1^{n_1} \cdots t_d^{n_d}$.
For each i we have the standard derivations $\partial_i \in \mathcal{V}_q$ satisfying

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Since A_q is noncommutative we also have inner derivations of form

$$ad\ t^n : p \mapsto t^n p - pt^n$$

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Result (Berman, Gau, Krylyuk 1996)

Each graded component $\mathrm{Der}(A_q)_n$ is spanned by **either** $ad\ t^n$ **or** $\{t^n \partial_1, \ldots, t^n \partial_d\}$ (depending on the multi-index n).

$\mathcal{V}_q$-modules

Result (W. Lin and S. Tan 2003)

There exists a family of functors

$$F_g^\alpha : \mathfrak{gl}_d\text{-mod} \to \mathrm{Der}(A_q)\text{-Mod}$$

$$F_g^\alpha : V \mapsto V \otimes \mathbb{A}_q$$

parametrized by $\alpha \in \mathbb{C}^d$ and group homomorphisms $g : \mathbb{Z}^d \to \mathbb{C}^*$.

$\mathcal{V}_q$-modules

Result (W. Lin and S. Tan 2003)

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The action of $\mathcal{V}_q$ on $V \otimes A_q$ can be written down explicilty.
These modules are all weight modules with respect to the Cartan-subalgebra spanned by the ∂_i.
These modules are also generically completely reducible.

$\mathcal{V}_q \ltimes A_q$-modules

Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ we have the corresponding **toroidal Lie algebra**

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \dots t_d^{\pm 1}].$$

In the classical work of E. Rao and others, it was shown that the classification of irreducible modules for $\hat{\mathfrak{g}}$ could be reduced to the classification of irreducible modules over the Lie algebra $\mathrm{Der}(A) \ltimes A$.

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Working towards a similar results in the quantum setting, E. Rao, P. Batra, and S. Sharma classified irreducible $\mathrm{Der}(A_q) \ltimes A_q$ modules with finite dimensional weight spaces when all q_{ij} are roots of unity. These modules all turned out to have form $V \otimes A_q$.

Part V

Rudakov modules

Rudakov's modules

Let $X = \mathbb{A}^n$, $A = k[x_1, \dots, x_n]$, $\mathfrak{L} := \mathcal{V} = \bigoplus_i k[x_1, \dots, x_n]\frac{\partial}{\partial x_i}$.
Define $\deg f\frac{\partial}{\partial x_i} = deg(f) - 1$, and
let $\mathfrak{L}(k) \subset \mathfrak{L}$ consist of derivations with no term of degree less than k.
This gives a filtration

$$\mathfrak{L} = \mathfrak{L}(-1) \supset \mathfrak{L}(0) \supset \mathfrak{L}(1) \supset \mathfrak{L}(2) \supset \cdots$$

Since $[\mathfrak{L}(i), \mathfrak{L}(k)] \subset \mathfrak{L}(i+k)$, each $\mathfrak{L}(k)$ is an ideal of $\mathfrak{L}(0)$.

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Since $[\mathfrak{L}(i), \mathfrak{L}(k)] \subset \mathfrak{L}(i+k)$, each $\mathfrak{L}(k)$ is an ideal of $\mathfrak{L}(0)$.

We have $\mathfrak{L}(0)/\mathfrak{L}(1) \simeq \mathfrak{gl}_s$ (with $E_{ij} \mapsto x_i\frac{\partial}{\partial x_j}$). Thus $\mathfrak{L}(0)$-modules with trivial $\mathfrak{L}(1)$-action corresponds to $\mathfrak{gl}_s$ modules, and we get a functor

$$\mathfrak{gl}_s\text{-mod} \to \mathfrak{L}\text{-Mod} \qquad U \mapsto \mathrm{Ind}_{\mathfrak{L}(0)}^{\mathfrak{L}} U.$$

Rudakov showed in 1974 that these induced modules are simple $\mathcal{V}$-modules except in a finite number of cases.

Generaliziations of Rudakov's work

Now let $X \subset \mathbb{A}^n$ be an affine smooth irreducible variety of dimension s.
Fix a point $P \in X$ and a simple $\mathfrak{gl}_s$-module U.
We define $\mathcal{V}(k) := \{\eta \in \mathcal{V} | \eta(A) \subset \mathfrak{m}_P^{k+1}\}$. Then we have

$$\mathcal{V} = \mathcal{V}(-1) \supset \mathcal{V}(0) \supset \mathcal{V}(1) \supset \mathcal{V}(2) \supset \cdots$$

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Proposition

We have $\mathcal{V}(0)/\mathcal{V}(k) \simeq \mathfrak{L}(0)/\mathfrak{L}(k)$ for each $k \geq 0$.

In particular, $\mathcal{V}(0)/\mathcal{V}(1) \simeq \mathfrak{gl}_s$.

Rudakov modules

Thus a $\mathfrak{gl}_s$-module U is a $\mathcal{V}(0)$-module. For $f \in A$, define

$$f \cdot u = f(P)u.$$

This action is compatible with the $\mathcal{V}$-action, so U is an $A\mathcal{V}(0)$-module.

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Definition

Given $U \in \mathfrak{gl}_s$-mod and $P \in X$ we define the corresponding **Rudakov module**

$$R_P(U) := A\#\mathcal{U}(\mathcal{V}) \bigotimes_{A\#\mathcal{U}(\mathcal{V}(0))} U.$$

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Theorem

When U is a simple $\mathfrak{gl}_s$-module, $R_P(U)$ is a simple $A\mathcal{V}$-module.

Gauge-Rudakov correspondance

Proposition

Let M be a gauge module and define $U := M/\mathfrak{m}_P M$. Then U is a finite dimensional $A\mathcal{V}(0)$-module, and there exists a natural pairing between the gauge module M and the Rudakov module $R_P(U^*)$.

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The dual of the projection $\pi : M \to U$ is $\pi^* : U^* \to M^*$. Write $\overline{\pi^*}$ for the canonical extension of π^* to $R_P(U^*)$.

Then the pairing is given by $\langle m, r\rangle = \overline{\pi^*}(r)(m)$. The pairing satisfies

$$\langle f \cdot m, r\rangle = \langle m, f \cdot r\rangle \quad \text{and} \quad \langle \eta \cdot m, r\rangle = -\langle m, \eta \cdot r\rangle$$

for all $f \in A$, $\eta \in \mathcal{V}$, $m \in M$, and $r \in R_P(U^*)$.

Thanks!