On CAR relations with orthogonal ranges

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The object

We study representations of $*$-algebras $\mathcal{A}_d$ generated by the pairs of elements, $a_1, \ldots, a_d$, satisfying

$$a_j^* a_j + a_j a_j^* = 1, \qquad a_j^* a_k = 0. \tag{1}$$

Such algebra arise, on one side, as a deformations of the Cuntz-Toeplitz algebras $\mathcal{O}_d$, and on the other side, as a Wick analogue of the classical CAR algebra.

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We study cases of $d = 2, 3$, but first recall the known case $d = 1$.

Case of $d = 1$

Consider $*$-algebra, generated by a, a^* with relation

$$a^* a = 1 - a a^*. \tag{2}$$

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Theorem

Irreducible representation of (2) are:

1. *Fock representation* π_F *in* $\mathbb{C}^2$, $\pi_F(a) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

2. *Continuous family* $\pi_{x,\phi}$ *in* $\mathbb{C}^2$,
$\pi_{x,\phi}(a) = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x} \\ \sqrt{1-x} & 0 \end{pmatrix}$,
where $\phi \in [0, 2\pi)$ *and* $0 < x < \frac{1}{2}$ *are fixed.*

3. *Continuous family* ρ_ϕ *in* $\mathbb{C}$, $\rho_\phi(a) = \frac{e^{i\phi}}{\sqrt{2}}, \quad \phi \in [0, 2\pi)$.

Wold decomposition

Using description of irreducible representations one can get an analog of the Wold decomposition for operator $A\colon H \to H$, satisfying (2). Namely, let $A = UC$, $C = (A^*A)^{\frac{1}{2}}$, U is partial isometry with $\ker U = \ker C = \ker A$, be the polar decompostion. Then the decomposition holds

$$H = H_F \oplus H_u$$

such that H_F and H_u are invariant with respect to A, A^*, and the restriction of A onto H_F is a multiple of the Fock representation, and the phase operator of the restriction of A onto H_u is unitary.

Case of $d = 2$. Relations

Consider operators in a Hilbert space H with relations

$$a_i^* a_i = 1 - a_i a_i^*, \quad i = 1, 2, \qquad a_1^* a_2 = 0. \tag{3}$$

Proposition

Let a_i, $i = 1, 2$ determine an irreducible representation of (3). Suppose that the unitary part H_u of the Wold decomposition of a_2 is non-zero. Then $a_1^2 = 0$.

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Assume that $a_1^2 = 0$. Then $H = \mathbb{C}^2 \otimes H_1$, and

$$a_1 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},$$

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and $a_2^* a_2 = 1 - a_2 a_2^*$ is equivalent to the following relations

$$\begin{aligned} A^* A &= I - AA^* - BB^*, \\ A^* B &= 0, \quad B^* B = I. \end{aligned} \tag{4}$$

Case of $d = 2$. Representations

Denote by S the one-sided shift operator in $\ell_2(\mathbb{Z}_+)$.

Proposition

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$\mathcal{H}_1 \simeq \ell_2(\mathbb{Z}_+) \otimes \mathcal{K}$ *for some Hilbert space $\mathcal{K}$ and*

$$B = S \otimes I, \quad A = (I - SS^*) \otimes \tilde{A},$$

where $\tilde{A}$ satisfies $\tilde{A}^*\tilde{A} + \tilde{A}\tilde{A}^* = I$ on $\mathcal{K}$.

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Combining the results above we get the complete list of unitarily inequivalent irreducible representations of $\mathcal{A}_2$.

Matrix decomposition for $\mathcal{A}_3$

Now consider representations of $*$-algebra $\mathcal{A}_3$:

$$\begin{aligned} a_j^* a_j + a_j a_j^* = I, \quad a_j^2 = 0, \qquad j = 1, 2, 3, \\ a_j^* a_k = 0, \qquad j \neq k, \end{aligned} \tag{5}$$

assuming also $a_j^2 = 0$, $j = 1, 2, 3$.

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assuming also $a_j^2 = 0$, $j = 1, 2, 3$.
The representations space can be decomposed as $H = H_0 \oplus H_0$ so that

$$a_1 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} A_1 & B_1 \\ 0 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} A_2 & B_2 \\ 0 & 0 \end{pmatrix}.$$

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Relations (5) are equivalent to the following set of relations

$$B_j^* B_k = \delta_{jk} I, \quad j, k = 1, 2,$$
$$A_1^* A_2 = 0, \quad A_j^* A_j + A_j A_j^* = I - B_j B_j^*, \quad A_j^2 = 0, \qquad j = 1, 2,$$
$$B_1^* A_1 = B_1^* A_1^* = B_2^* A_1 = 0, \quad B_2^* A_2 = B_2^* A_2^* = B_1^* A_2 = 0. \tag{6}$$

Fock representation

Let Λ_0 be the set of all finite multiindices

$$\Lambda_0 \ni \alpha = (\alpha_1, \dots, \alpha_m), \quad 1 \leq \alpha_j \leq 4, \quad j = 1, \dots, m; \; m = 0, 1, \dots,$$

(for $m = 0$ we assume single $\alpha_\emptyset \in \Lambda_0$) such that α does not contain pairs of neighbour indices $(3, 1)$, $(4, 2)$, $(3, 3)$ or $(4, 4)$.

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(for $m = 0$ we assume single $\alpha_\emptyset \in \Lambda_0$) such that α does not contain pairs of neighbour indices $(3,1)$, $(4,2)$, $(3,3)$ or $(4,4)$. Consider vectors e_α, $\alpha \in \Lambda_0$, and define the operators as follows

$$\begin{gathered}
A_1^* e_\emptyset = A_2^* e_\emptyset = B_1^* e_\emptyset = B_2^* e_\emptyset = 0, \\
B_1 e_\alpha = e_{\sigma_1(\alpha)}, \quad B_1^* e_\alpha = \delta_{1,\alpha_1} e_{\sigma(\alpha)}, \\
B_2 e_\alpha = e_{\sigma_2(\alpha)}, \quad B_2^* e_\alpha = \delta_{2,\alpha_1} e_{\sigma(\alpha)}, \\
A_1 e_\alpha = (1 - \delta_{3,\alpha_1})(1 - \delta_{1,\alpha_1}) e_{\sigma_3(\alpha)}, \quad A_1^* e_\alpha = \delta_{3,\alpha_1} e_{\sigma(\alpha)}, \\
A_2 e_\alpha = (1 - \delta_{4,\alpha_1})(1 - \delta_{2,\alpha_1}) e_{\sigma_4(\alpha)}, \quad A_2^* e_\alpha = \delta_{4,\alpha_1} e_{\sigma(\alpha)}.
\end{gathered}$$

Here, we use the standard notation

$$\begin{gathered}
\sigma(\alpha_1, \alpha_2, \ldots, \alpha_m) = (\alpha_2, \ldots, \alpha_m), \\
\sigma_j(\alpha_1, \ldots, \alpha_m) = (j, \alpha_1, \ldots, \alpha_m), \quad j = 1, 2, 3, 4.
\end{gathered}$$

Fock representation. Continued

Proposition

1. The set of vectors e_α, $\alpha \in \Lambda_0$, *is invariant for* A_i, A_i^*, B_i, B_i^*, $i = 1, 2$.
2. Taking e_α, $\alpha \in \Lambda_0$, *as an orthonormal basis, we get that* A_j^* *is adjoint to* A_j *and* B_j^* *is adjoint to* B_j, $j = 1, 2$.
3. The operators A_i, A_i^*, B_i, B_i^*, $i = 1, 2$, *satisfy* (6).
4. The operators A_i, A_i^*, B_i, B_i^*, $i = 1, 2$, *form an irreducible family.*

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Proposition

1. Operators $A_1 A_1^*$, $A_2 A_2^*$, $B_1 B_1^*$, $B_2 B_2^*$, *are projections with pairwise orthogonal ranges.*
2. Let for an irreducible representaion $P_0 = I - B_1 B_1^* - B_2 B_2^* - A_1 A_1^* - A_2 A_2^* \neq 0$. *Then* $\dim P_0 = 1$ *and the representtion is unitarily equivalent to the Fock representation.*

Non-Fock representations. Example

In what follows, we assume $B_1 B_1^* + B_2 B_2^* + A_1 A_1^* + A_2 A_2^* = I$.

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Let Λ be the set of all infinite multiindices

$$\Lambda \ni \alpha = (\alpha_1, \alpha_2, \dots), \qquad \alpha_j \in \{1, 2, 3, 4\},\ j = 1, 2, \dots,$$

for which, as above, we assume that α does not contain pairs of neighbour indices $(3, 1)$, $(4, 2)$, $(3, 3)$ or $(4, 4)$.

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Fix arbitrary $\lambda \in \Lambda$ and denote by Λ_λ the set of all multiindices $\alpha \in \Lambda$ which "have the same tail as λ up to a shift": $\alpha \in \Lambda_\lambda$ if and only if there exist numbers $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $\alpha_k = \lambda_{k+m}$ for all $k > n$.

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We construct a family of irreducible non-Fock representations parametrized by such sets Λ_λ, $\lambda \in \Lambda$.

Non-Fock representations. Formula

Consider a Hilbert space H_λ spanned by orthonormal basis e_α, $\alpha \in \Lambda_\lambda$ and define the operators

$$
\begin{gathered}
B_1 e_\alpha = e_{\sigma_1(\alpha)}, \quad B_2 e_\alpha = e_{\sigma_2(\alpha)}, \\
A_1 e_\alpha = (1 - \delta_{3,\alpha_1})(1 - \delta_{1,\alpha_1}) e_{\sigma_3(\alpha)}, \\
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Proposition

1. For the adjoint operators we have

$$\begin{aligned} B_1^* e_\alpha &= \delta_{1,\alpha_1} e_{\sigma(\alpha)}, \quad B_2^* e_\alpha = \delta_{2,\alpha_1} e_{\sigma(\alpha)}, \\ A_1^* e_\alpha &= \delta_{3,\alpha_1} e_{\sigma(\alpha)}, \quad A_2^* e_\alpha = \delta_{4,\alpha_1} e_{\sigma(\alpha)}. \end{aligned}$$

2. Operators A_i, A_i^*, B_i, B_i^*, $i = 1, 2$, satisfy (6).
3. Operators A_i, A_i^*, B_i, B_i^*, $i = 1, 2$, form an irreducible family.
4. Representations in H_λ and in $H_{\lambda'}$ given by (7) are unitary equivalent if and only if $\lambda' \in \Lambda_\lambda$.

General representations

To each finite multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda_0$, we associate the operator

$$P_\alpha = X_{\alpha_1} \ldots X_{\alpha_n} X^*_{\alpha_n} \ldots X^*_{\alpha_1}$$

where $X_1 = B_1, X_2 = B_2, X_3 = A_1, X_4 = A_2$, and set $P_\emptyset = I$.

Proposition

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Proposition

The family P_α is a commutative family of projections.
Also, we have

$$\begin{aligned} A_j P_\alpha &= (1 - \delta_{\alpha_1, j})(1 - \delta_{\alpha_1, j+2}) P_{\sigma_{j+2}(\alpha)} A_j, \\ A_j^* P_\alpha &= \delta_{\alpha_1, j} P_{\sigma\alpha} A_j^*, \quad \alpha \neq \emptyset, \\ B_j P_\alpha &= P_{\sigma_j(\alpha)} B_j. \end{aligned}$$

Sectral decomposition for the family P_α, $\alpha \in \Lambda_0$

Theorem

Let $B_1 B_1^* + B_2 B_2^* + A_1 A_1^* + A_2 A_2^* = I$. The spectral decomposition holds

$$P_\alpha = \int_\Lambda \delta_\alpha(\lambda)\, dE(\lambda), \quad \alpha \in \Lambda_0,$$

where $\delta_{(\alpha_1,\dots,\alpha_n)}((\lambda_1,\dots,\lambda_n,\lambda_{n+1}\dots)) = \delta_{\alpha_1,\lambda_1}\cdots\delta_{\alpha_n,\lambda_n}$, and $E(\cdot)$ is a resolution of the identity on the cylinder σ-algebra in Λ.

Formula for representations (commutative model)

Theorem

Any representation acts as follows.

$$H = \int_{\Lambda}^{\oplus} H_\lambda \, d\mu(\lambda),$$

$$(B_j f)(\lambda) = V_j(\lambda)\delta_{\lambda_1, j}\sqrt{\frac{d(\delta_{j,\lambda_1} \otimes \mu(\sigma(\lambda)))}{d\mu(\lambda)}} f(\sigma(\lambda)),$$

$$(A_j f)(\lambda) = U_j(\lambda)\delta_{\lambda_1, j+2}\sqrt{\frac{d(\delta_{j+2,\lambda_1} \otimes \mu(\sigma(\lambda)))}{d\mu(\lambda)}} f(\sigma(\lambda)), \quad j = 1, 2,$$

Here, μ is a measure on the cylinder σ-algebra in Λ invariant w.r.t. the transformations involved, V_1, V_2, U_1, U_2 are unitary measurable operator-valued functions, and H_λ is a measurable field of Hilbert spaces, for which $\dim H_{\sigma(\lambda)} = \dim H_\lambda$ μ-a.e.