On representations of finite $W$-algebras

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1. Introduction

- A finite $W$-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.

- A finite $W$-algebra is a generalization of the universal enveloping algebra $U(\mathfrak{g})$. For $e = 0$ it coincides with $U(\mathfrak{g})$.

- Finite $W$-algebra is a quantization of the Poisson algebra of functions on the Slodowy (i.e. transversal) slice at $e$ to the orbit $Ad(G)e$, where $\mathfrak{g} = \text{Lie}(G)$.

- Due to recent results of I. Losev, A. Premet and others, finite $W$-algebras play a very important role in description of primitive ideals.

- Finite $W$-algebras for semi-simple Lie algebras were introduced by A. Premet.

**Definition.** An element $e \in \mathfrak{g}$ is *nilpotent* if $ad_e$ is a nilpotent endomorphism of $\mathfrak{g}$. A nilpotent element $e \in \mathfrak{g}$ is *regular* nilpotent if the centralizer $\mathfrak{g}^e$ attains the minimal dimension, which is equal to $\text{rank}\mathfrak{g}$.

**Theorem.** *(B. Kostant, 1978)* For a reductive Lie algebra $\mathfrak{g}$ and a regular nilpotent element $e \in \mathfrak{g}$, the finite $W$-algebra coincides with the center of $U(\mathfrak{g})$.

- This theorem does not hold for Lie superalgebras, since the finite $W$-algebra has a non-trivial odd part, while the center of $U(\mathfrak{g})$ is even.

*(V. Kac, M. Gorelik, A. Sergeev)*
• J. Brown, J. Brundan and S. Goodwin described principal finite $W$-algebra $W_{m|n}$ for $\mathfrak{gl}(m|n)$ associated with even regular (i.e. principal) nilpotent $e$ as truncation of shifted super-Yangian of $\mathfrak{gl}(1|1)$.

• They proved that simple $W_{m|n}$-modules are finite-dimensional and classified them by highest weight theory using the triangular decomposition

$$W_{m|n} = W^-_{m|n} W^0_{m|n} W^+_{m|n}$$

2. **The Queer Lie Superalgebra** $\mathfrak{g} = \mathcal{Q}(n)$

- Equip $\mathbb{C}^{n|n}$ with the odd operator $\zeta$ such that $\zeta^2 = -\text{Id}$.

Let $\mathcal{Q}(n)$ be the centralizer of $\zeta$ in the Lie superalgebra $\mathfrak{gl}(n|n)$. Then

$$\mathcal{Q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \text{ are } n \times n \text{ matrices} \right\}$$

- Supercommutator: $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$.

- Standard bases in $A$ and $B$ respectively:

$$e_{i,j} = \begin{pmatrix} E_{ij} & 0 \\ 0 & 0 \end{pmatrix}, \quad f_{i,j} = \begin{pmatrix} 0 & E_{ij} \\ E_{ij} & 0 \end{pmatrix}$$
• $\mathfrak{g} = Q(n)$ admits an odd non-degenerate $\mathfrak{g}$-invariant super-symmetric bilinear form

$$(X|Y) := otr(XY) \text{ for } X, Y \in Q(n)$$

$$otr \begin{pmatrix} A & B \\ B & A \end{pmatrix} = trB$$

• $\mathfrak{g}^* \cong \Pi(\mathfrak{g})$, where $\Pi$ is the change of parity functor.
• We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal $A$ and $B$.

$$\mathfrak{h} = \text{Span}\{e_{i,i} \mid f_{i,i}\}, \quad i = 1, \ldots, n$$

• $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) is the nilpotent subalgebra consisting of matrices with strictly upper triangular (respectively, low triangular) $A$ and $B$.

• The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

Set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. 

3. The finite $W$-algebra for $Q(n)$

- We define the finite $W$-algebra associated with the regular even nilpotent element $\chi$ in the coadjoint representation of $Q(n)$.
- Choose $\chi \in \mathfrak{g}^*$ such that

$$\chi(f_{i,j}) = 0, \quad \chi(e_{i,j}) = \delta_{i,j+1}. $$

**Remark.** Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$ (odd). Then $\chi(x) = (x|E)$ for $x \in \mathfrak{g}$.

$$E = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
$$

$\chi$ is regular nilpotent $\iff$ $E$ has a single Jordan block.
Let $I_{\chi}$ be the left ideal in $U(\mathfrak{g})$ generated by $x - \chi(x)$ for all $x \in \mathfrak{n}^-$, and

$\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_{\chi}$ be the natural projection.

**Definition.** The finite $W$-algebra associated with $\chi$ is

$$W := \{ \pi(y) \in U(\mathfrak{g})/I_{\chi} | \text{ad}(x)y \in I_{\chi} \text{ for all } x \in \mathfrak{n}^- \}.$$ 

$$\pi(y_1)\pi(y_2) = \pi(y_1y_2)$$

- We identify $U(\mathfrak{g})/I_{\chi}$ with $U(\mathfrak{b})$, then $W$ is a subalgebra of $U(\mathfrak{b})$. 
**Definition.** The *Harish-Chandra homomorphism* is the natural projection
\[ \vartheta : U(b) \to U(h) \]
with the kernel \( n^+U(b) \).

\[ \vartheta : W \longrightarrow U(h) \]
is injective.

We consider \( W \) as a subalgebra of \( U(h) \).
4. The structure of $W$-algebra

- The Cartan subalgebra of $\mathfrak{g} = Q(n)$ is

$$\mathfrak{h} = \text{Span}\{e_{i,i} \mid f_{i,i}\}.$$

$$[f_{i,i}, f_{j,j}] = 0 \text{ if } i \neq j, \quad [f_{i,i}, f_{i,i}] = 2e_{i,i}.$$

Set $\xi_i = (-1)^{i+1}f_{i,i}, \quad x_i = \xi_i^2 = e_{i,i}$. Then

- $U(\mathfrak{h}) = \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i)_{i < j \leq n}$.

- The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \ldots, x_n]$.

- The center of $W$ coincides with $W \cap \mathbb{C}[x_1, \ldots, x_n]$. It is the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism. (Adv. Math., 2016).

- The center of $U(\mathfrak{g})$ is generated by $Q$-symmetric polynomials (A. Sergeev).
• We define the following set of generators of $W$:
  
  $n$ odd generators $\phi_k$ and $n$ even generators $z_k$.

Set

$$\phi_0 = \sum_{i=1}^{n} \xi_i, \quad \phi_k = T^k(\phi_0), \quad k \geq 0,$$

where the matrix of $T$ in the standard basis $\xi_1, \ldots, \xi_n$ has 0 on the diagonal and

$$t_{ij} = \begin{cases} 
  x_j & \text{if } i < j, \\
  -x_j & \text{if } i > j.
\end{cases}$$
For even $k \geq 0$ set 
\[
z_k := [\phi_0, \phi_k] \in \text{center of } W
\]

For odd $k < n$ set 
\[
z_k := \left[ \sum_{i_1 \geq i_2 \geq \ldots \geq i_{k+1}} (x_{i_1} + (-1)^k \xi_{i_1}) \ldots (x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}}) \right]_{\text{even}},
\]

Then 
\[
[\phi_i, \phi_j] = \begin{cases} 
(-1)^i z_{i+j} & \text{if } i + j \text{ is even} \\
0 & \text{if } i + j \text{ is odd}
\end{cases}
\]

• Elements $z_0, \ldots, z_{n-1}$ are algebraically independent in $W$ and they commute with each other. Together with $\phi_0, \ldots, \phi_{n-1}$ they form a complete set of generators in $W$. 
5. IRREDUCIBLE REPRESENTATIONS OF $W$

• We proved that all simple $W$-modules are finite-dimensional (Adv. Math., 2016).
Now we give a classification of simple $W$-modules.

Restriction from $U(\mathfrak{h})$ to $W$.

Definition. Let $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$. Then $s$ regular if $s_i \neq 0$ for all $i \leq n$ and typical if $s_i + s_j \neq 0$ for all $i \neq j \leq n$.

• All irreducible representations of $U(\mathfrak{h})$ are enumerated by $s \in \mathbb{C}^n$ up to change of parity.

Let $V$ be an irreducible representation, then every $x_i$ acts by scalar $s_i \text{Id}$.

Let $I_s$ be the ideal in $U(\mathfrak{h})$ generated by $x_i - s_i$.

Then the quotient algebra $U(\mathfrak{h})/I_s$ is isomorphic to the Clifford algebra $C_s$ associated with the quadratic form $B_s$:

$$C_s = \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i \xi_j + \xi_j \xi_i - 2\delta_{i,j}s_i),$$

and $V$ is a simple $C_s$-module.
Let $m(s)$ be the number of non-zero coordinates of $s$. Then

- $C_s$ has **one** simple $\mathbb{Z}_2$-graded module $V(s)$ for **odd** $m(s)$,

and **two** simple modules $V(s)$ and $\Pi V(s)$ for **even** $m(s)$.

In the case when $s$ is regular, the form $B_s$ is non-degenerate and the dimension of $V(s)$ equals $2^k$, where $k = \lceil n/2 \rceil$.

- We denote by the same symbol $V(s)$ the restriction to $W$. 
Proposition 1. Let $S$ be a simple $W$-module. Then $S$ is a simple constituent of $V(s)$ for some $s \in \mathbb{C}^n$.

Proposition 2. If $s$ is typical, then $V(s)$ is a simple $W$-module.

Sketch of proof. Consider $U(\mathfrak{h})$ as a free $U(\mathfrak{h}_0)$-module. Note that all $\phi_i$ belong to the free submodule generated by $\xi_1, \ldots, \xi_n$, which is equipped with $U(\mathfrak{h}_0)$-valued bilinear symmetric form

$$B(x, y) = [x, y].$$

Let $\Gamma$ denotes the Gramm matrix $B(\phi_i, \phi_j)$. Then

$$\det \Gamma = cp^2 x_1 \ldots x_n,$$

where $p(x_1, \ldots, x_n) := \prod_{i<j}(x_i + x_j)$ and $c$ is a non-zero constant.

- Assume that $s$ is regular, i.e. $s_i \neq 0$ for all $i = 1, \ldots, n$. Since after specialization $x_i \mapsto s_i$ we have $\det \Gamma(s) \neq 0$, we have an isomorphism of superalgebras $C_s$ and $W/(W \cap I_s)$.

- If $s$ is typical non-regular, then there is exactly one $i$ such that $s_i = 0$, and the statement follows from the regular case for $n - 1$. 
Proposition 3. Let $s' = \sigma(s)$ for some permutation of coordinates.

(a) If $s$ is typical, then $V(s)$ is isomorphic to $V(s')$ as a $W$-module.

(b) If $s$ is arbitrary, then $V(s)$ as a $W$-module has the same Jorgan-Hölder series as $V(s')$ or $\Pi V(s')$ up to permutation.
6. **Simple $W$-modules for $Q(2)$**

- The action of $U(\mathfrak{h})$ in $V(s_1, s_2)$ is given by the following formulas in a suitable basis:

$$
\xi_1 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} \\ \sqrt{s_1} & 0 \end{pmatrix}, \quad \xi_2 \mapsto \begin{pmatrix} 0 & \sqrt{s_2 i} \\ -\sqrt{s_2 i} & 0 \end{pmatrix}.
$$

- The generators of $W$ are

$$
\phi_0 = \xi_1 + \xi_2, \quad \phi_1 = x_2 \xi_1 - x_1 \xi_2,
$$

$$
z_0 = x_1 + x_2, \quad z_1 = x_1 x_2 - \xi_1 \xi_2.
$$

and they act by

\begin{align*}
(1) \quad \phi_0 & \mapsto \begin{pmatrix} 0 & \sqrt{s_1 + \sqrt{s_2 i}} \\ \sqrt{s_1 - \sqrt{s_2 i}} & 0 \end{pmatrix}, \quad \phi_1 & \mapsto \sqrt{s_1 s_2} \begin{pmatrix} 0 & \sqrt{s_2 - \sqrt{s_1 i}} \\ \sqrt{s_2 + \sqrt{s_1 i}} & 0 \end{pmatrix}, \\
(2) \quad z_0 & \mapsto s_1 + s_2, \quad z_1 \mapsto \begin{pmatrix} s_1 s_2 + \sqrt{s_1 s_2 i} & 0 \\ 0 & s_1 s_2 - \sqrt{s_1 s_2 i} \end{pmatrix}.
\end{align*}
• $V(s)$ is simple as $W$-module if and only if $s_1 \neq -s_2$.

Let $s_1 = -s_2$. If $s_1 = s_2 = 0$, then $V(s) \cong \mathbb{C} \oplus \Pi(\mathbb{C})$.

If $s_1 \neq 0$, we choose $\sqrt{s_1}, \sqrt{s_2}$ so that $\sqrt{s_2} = \sqrt{s_1}i$. Then the following exact sequence follows from (1) and (2):

$$0 \to \Pi \Gamma_{-s_1^2 + s_1} \to V(s) \to \Gamma_{-s_1^2 - s_1} \to 0,$$

where $\Gamma_t$ is one-dimensional simple module on which $\phi_0, \phi_1$ and $z_0$ act by zero and $z_1$ acts by the scalar $t$.

Lemma 1. If $n = 2$, then every simple $W$-module is isomorphic to one of the following

(1) $V(s_1, s_2)$ or $\Pi V(s_1, s_2)$ for $s_1 \neq -s_2, s_1, s_2 \neq 0$;

(2) $V(s, 0)$ if $s \neq 0$;

(3) $\Gamma_t$ or $\Pi \Gamma_t$. 

7. General construction of simple $W$-modules

Let $W$ be the finite $W$-algebra for $Q(n)$.

Let $i + j = n$. There is natural embedding of the Lie superalgebras:

$$Q(i) \oplus Q(j) \hookrightarrow Q(n).$$

This induces the isomorphism

$$U(h) \simeq U(h_i) \otimes U(h_j),$$

where $h_r$ denotes the Cartan subalgebra of $Q(r)$.

**Lemma 2.** Let $i + j = n$. Then $W$ is a subalgebra in the tensor product $W^i \otimes W^j$, where $W^r \subset U(h_r)$ denotes the $W$-algebra for $Q(r)$.

**Corollary.** If $i_1 + \cdots + i_p = n$, then $W$ is a subalgebra in $W^{i_1} \otimes \cdots \otimes W^{i_p}$. 
Let \( r, p, q \in \mathbb{N} \cup \{0\} \) and \( r + 2p + q = n \),
\[
t = (t_1, \ldots, t_p) \in \mathbb{C}^p, \ t_1, \ldots, t_p \neq 0,
\]
\[
\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{C}^q, \ \lambda_1, \ldots, \lambda_q \neq 0, \text{ such that } \lambda_i + \lambda_j \neq 0 \text{ for any } 1 \leq i \neq j \leq q.
\]
We have an embedding
\[
W \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q.
\]
Set
\[
S(t, \lambda) := \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p} \boxtimes V(\lambda).
\]

**Theorem 2.** (P.–S., 2019)

1. \( S(t, \lambda) \) is a simple \( W \)-module;
2. Every simple \( W \)-module is isomorphic to \( S(t, \lambda) \) up to change of parity.

**Idea of proof.** Use Propositions (1)–(3).

**Proposition 4.**

Two simple modules \( S(t, \lambda) \) and \( S(t', \lambda') \) are isomorphic if and only if \( t' = \sigma(t) \) and \( \lambda' = \tau(\lambda) \) for some \( \sigma \in S_p \) and \( \tau \in S_q \).
8. Central characters

• The center $Z$ of $W$ coincides with the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism. It is generated by the $Q$-symmetric polynomials (A. Sergeev):

$$p_k = x_1^{2k+1} + \cdots + x_n^{2k+1} \text{ for all } k \in \mathbb{N}$$

• Every $s \in \mathbb{C}^n$ defines the central character $\chi_s : Z \to \mathbb{C}$. Theorem 2–(2) implies that every simple $W$-module admits central character $\chi_s$ for some $s$.

**Definition.** For every $s = (s_1, \ldots, s_n)$ we define the core $c(s) = (s_{i_1}, \ldots, s_{i_m})$ as a subsequence obtained from $s$ by removing all $s_j = 0$ and all pairs $(s_i, s_j)$ such that $s_i + s_j = 0$. The core is well defined up to permutation.

**Example.** Let $s = (1, 0, 3, -1, -1)$, then $c(s) = (3, -1)$.

**Lemma 3.** Let $s, s' \in \mathbb{C}^n$. Then $\chi_s = \chi_{s'}$ if and only if $s$ and $s'$ have the same core (up to permutation) (A. Sergeev).

Thus the core depends only on the central character $\chi_s$, we denote it $c(\chi)$. 
The category $W - \text{mod}$ of finite dimensional $W$-modules decomposes into direct sum $\bigoplus W^{\chi} - \text{mod}$, where $W^{\chi} - \text{mod}$ is the full subcategory of modules admitting generalized central character $\chi$.

**Lemma 4.** A simple $W$-module $S$ belongs to $W^{\chi} - \text{mod}$ if and only if it is isomorphic to $S(t, \lambda)$ with $\lambda = c(\chi)$.

**Proof.** We have to compute the central character of $S(t, \lambda)$. For a $Q$-symmetric polynomial $p_k = x_1^{2k+1} + \cdots + x_n^{2k+1}$ we have $p_k(t, \lambda) = \lambda_1^{2k+1} + \cdots + \lambda_q^{2k+1}$. Since $p_k$ generate the center of $W$ the statement follows.

We plan to describe blocks in the category of finite dimensional $W$-modules.

We did this for $Q(2)$. 

9. Connection with super-Yangians

Super-Yangian $Y(Q(n))$ was introduced by M. Nazarov. *(Lecture Notes in Math. 1992)*

- $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators $T_{i,j}^{(m)}$ where $m = 1, 2, \ldots$ and $i, j = \pm 1, \pm 2, \ldots, \pm n$.

- The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j)$$

where $p(i) = 0$ if $i > 0$ and $p(i) = 1$ if $i < 0$. 
• To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T^{(1)}_{i,j} u^{-1} + T^{(2)}_{i,j} u^{-2} + \ldots$$

$$(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u))$$

$$- (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}$$

(1)

$$T_{i,j}(-u) = T_{-i,-j}(u)$$

(2)

• We proved that in the regular case the finite $W$-algebra for $Q(n)$ is a quotient of the super-Yangian of $Q(1)$ (Adv. Math., 2016).

• We generalized this result for $Q(nl)$ when the corresponding nilpotent element has Jordan blocks each of size $l$. We proved that the finite $W$-algebra is a quotient of the super-Yangian of $Q(n)$ (J. Math. Phys., 2017).

Our results should have applications to classification of simple modules for super-Yangians of type $Q$. 
10. **Whittaker coinvariants (work in progress)**

- J. Brundan and S. Goodwin studied the *Whittaker coinvariants functor*: an exact functor from category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ to a certain category of finite-dimensional modules over $W_{m|n}$. (*Adv. Math. 2019*).

- Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, and $e \in \mathfrak{g}_0$ be regular nilpotent.

Consider a good $\mathbb{Z}$-grading for $e$: $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, $e \in \mathfrak{g}_1$. Let

$$p := \bigoplus_{i \geq 0} \mathfrak{g}_i, \ h := \mathfrak{g}_0, \ m := \bigoplus_{i < 0} \mathfrak{g}_i.$$ 

Let $\chi \in \mathfrak{g}^*$ be given by $\chi(x) := (x|e)$. 

Let $I_\chi$ be the right ideal of $U(\mathfrak{g})$ generated by $x - \chi(x)$ for all $x \in \mathfrak{m}$.

Define the principal $W$-algebra by

$$W := \{ u \in U(\mathfrak{p}) \mid uI_\chi \subseteq I_\chi \}.$$ 

Let $M$ be a left $\mathfrak{g}$-module. Then there is a well-defined left action of $W$ on $H_0(M) := M/I_\chi M$.

This gives the Whittaker coinvariants functor

$$H_0 : U(\mathfrak{g}) \text{- mod} \longrightarrow W \text{- mod}.$$ 

We plan to study the Whittaker coinvariants functor for $\mathfrak{g} = Q(n)$. 
11. REFERENCES


