On representations of finite W-algebras

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1. Introduction

• A *finite W-algebra* is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ is a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.

• A finite W-algebra is a generalization of the universal enveloping algebra $U(\mathfrak{g})$. For $e = 0$ it coincides with $U(\mathfrak{g})$.

• Finite W-algebra is a quantization of the Poisson algebra of functions on the Slodowy (i.e. transversal) slice at e to the orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$.

• Due to recent results of I. Losev, A. Premet and others, finite W-algebras play a very important role in description of primitive ideals.

• Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet.

• Finite W-algebras for Lie algebras and superalgebras have been studied by mathematicians and physicists: L. Fehér, C. Briot, E. Ragoucy, P. Sorba, V. G. Kac, A. Premet, I. Losev, V. Ginzburg, W. L. Gan, J. Brundan, A. Kleshchev, J. Brown, S. Goodwin, W. Wang, L. Zhao, Y. Zeng, B. Shu, Y. Peng.

Definition. An element $e \in \mathfrak{g}$ is *nilpotent* if $\mathrm{ad}e$ is a nilpotent endomorphism of $\mathfrak{g}$.
A nilpotent element $e \in \mathfrak{g}$ is *regular* nilpotent if the centralizer $\mathfrak{g}^e$ attains the minimal dimension, which is equal to $rank\mathfrak{g}$.

Theorem. *(B. Kostant, 1978)* For a reductive Lie algebra $\mathfrak{g}$ and a *regular* nilpotent element $e \in \mathfrak{g}$, the finite W-algebra coincides with the center of $U(\mathfrak{g})$.

- This theorem does not hold for Lie superalgebras, since the finite W-algebra has a non-trivial odd part, while the center of $U(\mathfrak{g})$ is even.

(V. Kac, M. Gorelik, A. Sergeev)

• J. Brown, J. Brundan and S. Goodwin described *principal* finite W-algebra $W_{m|n}$ for $\mathfrak{gl}(m|n)$ associated with even *regular* (i.e. principal) nilpotent e as truncation of shifted super-Yangian of $\mathfrak{gl}(1|1)$.

• They proved that simple $W_{m|n}$-modules are finite-dimensional and classified them by highest weight theory using the triangular decomposition

$$W_{m|n} = W^{-}_{m|n} W^{0}_{m|n} W^{+}_{m|n}$$

(Algebra Numb. Theory, 2013).

2. The queer Lie superalgebra $\mathfrak{g} = \mathbf{Q(n)}$

• Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 =$ -Id.

Let $Q(n)$ be the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$. Then

$$Q(n) = \{\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A, B \text{ are } n \times n \text{ matrices}\}$$

• Supercommutator: $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.

• Standard bases in A and B respectively:

$$e_{i,j} = \left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j} = \left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right)$$

- $\mathfrak{g} = Q(n)$ admits an **odd** non-degenerate $\mathfrak{g}$-invariant super-symmetric bilinear form

$$(X|Y) := otr(XY) \text{ for } X, Y \in Q(n)$$

$$otr\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = trB$$

- $\mathfrak{g}^* \cong \Pi(\mathfrak{g})$, where Π is the change of parity functor.

• We fix the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ to be the set of matrices with diagonal A and B.

$$\mathfrak{h} = \mathrm{Span}\{e_{i,i} \mid f_{i,i}\}, \qquad i = 1, \dots, n$$

• $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) is the nilpotent subalgebra consisting of matrices with strictly upper triangular (respectively, low triangular) A and B.

• The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

Set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$.

3. The finite W-algebra for $Q(n)$

- We define the finite W-algebra associated with the *regular* even nilpotent element χ in the coadjoint representation of $Q(n)$.
- Choose $\chi \in \mathfrak{g}^*$ such that

$$\chi(f_{i,j}) = 0, \quad \chi(e_{i,j}) = \delta_{i,j+1}.$$

Remark. Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$ (**odd**). Then $\chi(x) = (x|E)$ for $x \in \mathfrak{g}$.

$$E = \left(\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right)$$

χ is *regular* nilpotent $\Longleftrightarrow$ E has a *single* Jordan block

Let I_χ be the left ideal in $U(\mathfrak{g})$ generated by $x - \chi(x)$ for all $x \in \mathfrak{n}^-$, and

$\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ be the natural projection.

Definition. *The finite W-algebra* associated with χ is

$$W := \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \operatorname{ad}(x)y \in I_\chi \text{ for all } x \in \mathfrak{n}^-\}.$$

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

- We identify $U(\mathfrak{g})/I_\chi$ with $U(\mathfrak{b})$, then W is a subalgebra of $U(\mathfrak{b})$.

Definition. The *Harish-Chandra homomorphism* is the natural projection

$$\vartheta : U(\mathfrak{b}) \to U(\mathfrak{h})$$

with the kernel $\mathfrak{n}^+ U(\mathfrak{b})$.

Theorem 1. *(P.–S., Adv. Math., 2016)* The restriction

$$\vartheta : W \longrightarrow U(\mathfrak{h})$$

is injective.

We consider W as a subalgebra of $U(\mathfrak{h})$.

4. The structure of W-algebra

- The Cartan subalgebra of $\mathfrak{g} = Q(n)$ is

$$\mathfrak{h} = \mathrm{Span}\{e_{i,i} \mid f_{i,i}\}.$$

$$[f_{i,i}, f_{j,j}] = 0 \text{ if } i \neq j, [f_{i,i}, f_{i,i}] = 2e_{i,i}.$$

Set $\xi_i = (-1)^{i+1} f_{i,i}$, $x_i = \xi_i^2 = e_{i,i}$. Then

- $U(\mathfrak{h}) = \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i)_{i<j\leq n}$.

- The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1, \ldots, x_n]$.

- The *center* of W coincides with $W \bigcap \mathbb{C}[x_1, \ldots, x_n]$.
  It is the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism.
(Adv. Math., 2016).

- The center of $U(\mathfrak{g})$ is generated by Q-symmetric polynomials *(A. Sergeev)*.

- We define the following set of generators of W:
  n **odd** generators ϕ_k and n **even** generators z_k.

Set

$$\phi_0 = \sum_{i=1}^{n} \xi_i, \quad \phi_k = T^k(\phi_0), \quad k \geq 0,$$

where the matrix of T in the standard basis $\xi_1, \ldots, \xi_n$ has 0 on the diagonal and

$$t_{ij} = \begin{cases} x_j & if \quad i < j, \\ -x_j & if \quad i > j. \end{cases}$$

For *even* $k \geq 0$ set

$$z_k := [\phi_0, \phi_k] \in \text{ center of } W$$

For *odd* $k < n$ set

$$z_k := [\sum_{i_1 \geq i_2 \geq \ldots \geq i_{k+1}} (x_{i_1} + (-1)^k \xi_{i_1}) \ldots (x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}})]_{even},$$

Then

$$[\phi_i, \phi_j] = \begin{cases} (-1)^i z_{i+j} \text{ if } i+j \text{ is even} \\ 0 \text{ if } i+j \text{ is odd} \end{cases}$$

- Elements $z_0, \ldots, z_{n-1}$ are algebraically independent in W and they commute with each other. Together with $\phi_0, \ldots, \phi_{n-1}$ they form a complete set of generators in W.

5. Irreducible representations of W

• We proved that all simple W-modules are finite-dimensional *(Adv. Math., 2016)*. Now we give a classification of simple W-modules.

Restriction from $U(\mathfrak{h})$ to W.

Definition. Let $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$. Then $\mathbf{s}$ *regular* if $s_i \neq 0$ for all $i \leq n$ and *typical* if $s_i + s_j \neq 0$ for all $i \neq j \leq n$.

• All irreducible representations of $U(\mathfrak{h})$ are enumerated by $\mathbf{s} \in \mathbb{C}^n$ up to change of parity.

Let V be an irreducible representation, then every x_i acts by scalar $s_i\mathrm{Id}$.

Let $I_\mathbf{s}$ be the ideal in $U(\mathfrak{h})$ generated by $x_i - s_i$.

Then the quotient algebra $U(\mathfrak{h})/I_\mathbf{s}$ is isomorphic to the Clifford algebra $C_\mathbf{s}$ associated with the quadratic form $B_\mathbf{s}$:

$$C_\mathbf{s} = \mathbb{C}[\xi_1, \ldots, \xi_n]/(\xi_i\xi_j + \xi_j\xi_i - 2\delta_{i,j}s_i),$$

and V is a simple $C_\mathbf{s}$-module.

Let $m(\mathbf{s})$ be the number of non-zero coordinates of $\mathbf{s}$. Then

• $C_{\mathbf{s}}$ has **one** simple $\mathbb{Z}_2$-graded module $V(\mathbf{s})$ for **odd** $m(s)$,

and **two** simple modules $V(\mathbf{s})$ and $\Pi V(\mathbf{s})$ for **even** $m(s)$.

In the case when $\mathbf{s}$ is regular, the form $B_{\mathbf{s}}$ is non-degenerate and the dimension of $V(\mathbf{s})$ equals 2^k, where $k = \lceil n/2 \rceil$.

• We denote by the same symbol $V(\mathbf{s})$ the restriction to W.

Proposition 1. Let S be a simple W-module. Then S is a simple constituent of $V(\mathbf{s})$ for some $\mathbf{s} \in \mathbb{C}^n$.

Proposition 2. If $\mathbf{s}$ is **typical**, then $V(\mathbf{s})$ is a simple W-module.

Sketch of proof. Consider $U(\mathfrak{h})$ as a free $U(\mathfrak{h}_{\bar{0}})$-module. Note that all ϕ_i belong to the free submodule generated by $\xi_1, \dots, \xi_n$, which is equipped with $U(\mathfrak{h}_{\bar{0}})$-valued bilinear symmetric form

$$B(x, y) = [x, y].$$

Let Γ denotes the Gramm matrix $B(\phi_i, \phi_j)$. Then

$$\det \Gamma = cp^2 x_1 \dots x_n,$$

where $p(x_1, \dots, x_n) := \prod_{i<j}(x_i + x_j)$ and c is a non-zero constant.

• Assume that $\mathbf{s}$ is regular, i.e. $s_i \neq 0$ for all $i = 1, \dots, n$. Since after specialization $x_i \mapsto s_i$ we have $\det \Gamma(\mathbf{s}) \neq 0$, we have an isomorphism of superalgebras $C_{\mathbf{s}}$ and $W/(W \cap I_{\mathbf{s}})$.

• If $\mathbf{s}$ is typical non-regular, then there is exactly one i such that $s_i = 0$, and the statement follows from the regular case for $n - 1$.

Proposition 3. Let $\mathbf{s}' = \sigma(\mathbf{s})$ for some permutation of coordinates.

(a) If $\mathbf{s}$ is typical, then $V(\mathbf{s})$ is isomorphic to $V(\mathbf{s}')$ as a W-module.

(b) If $\mathbf{s}$ is arbitrary, then $V(\mathbf{s})$ as a W-module has the same Jorgan-Hölder series as $V(\mathbf{s}')$ or $\Pi V(\mathbf{s}')$ up to permutation.

6. Simple W-modules for $Q(2)$

- The action of $U(\mathfrak{h})$ in $V(s_1, s_2)$ is given by the following formulas in a suitable basis:

$$\xi_1 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} \\ \sqrt{s_1} & 0 \end{pmatrix}, \quad \xi_2 \mapsto \begin{pmatrix} 0 & \sqrt{s_2}\mathbf{i} \\ -\sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}.$$

- The generators of W are

$$\phi_0 = \xi_1 + \xi_2,\ \phi_1 = x_2\xi_1 - x_1\xi_2,$$
$$z_0 = x_1 + x_2,\ z_1 = x_1x_2 - \xi_1\xi_2.$$

and they act by

$$(1) \qquad \phi_0 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} + \sqrt{s_2}\mathbf{i} \\ \sqrt{s_1} - \sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}, \quad \phi_1 \mapsto \sqrt{s_1 s_2} \begin{pmatrix} 0 & \sqrt{s_2} - \sqrt{s_1}\mathbf{i} \\ \sqrt{s_2} + \sqrt{s_1}\mathbf{i} & 0 \end{pmatrix},$$

$$(2) \qquad z_0 \mapsto s_1 + s_2, \quad z_1 \mapsto \begin{pmatrix} s_1s_2 + \sqrt{s_1s_2}\mathbf{i} & 0 \\ 0 & s_1s_2 - \sqrt{s_1s_2}\mathbf{i} \end{pmatrix}.$$

• $V(\mathbf{s})$ is simple as W-module if and only if $s_1 \neq -s_2$.

Let $s_1 = -s_2$. If $s_1 = s_2 = 0$, then $V(\mathbf{s}) \cong \mathbb{C} \oplus \Pi(\mathbb{C})$.

If $s_1 \neq 0$, we choose $\sqrt{s_1}, \sqrt{s_2}$ so that $\sqrt{s_2} = \sqrt{s_1}\mathbf{i}$. Then the following exact sequence follows from (1) and (2):

$$0 \to \Pi\Gamma_{-s_1^2+s_1} \to V(\mathbf{s}) \to \Gamma_{-s_1^2-s_1} \to 0,$$

where Γ_t is one-dimensional simple module on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t.

Lemma 1. If $n = 2$, then every simple W-module is isomorphic to one of the following

(1) $V(s_1, s_2)$ or $\Pi V(s_1, s_2)$ for $s_1 \neq -s_2$, $s_1, s_2 \neq 0$;

(2) $V(s, 0)$ if $s \neq 0$;

(3) Γ_t or $\Pi\Gamma_t$.

7. General construction of simple W-modules

Let W be the finite W-algebra for $Q(n)$.

Let $i+j=n$. There is natural embedding of the Lie superalgebras:

$$Q(i) \oplus Q(j) \hookrightarrow Q(n).$$

This induces the isomorphism

$$U(\mathfrak{h}) \simeq U(\mathfrak{h}_i) \otimes U(\mathfrak{h}_j),$$

where $\mathfrak{h}_r$ denotes the Cartan subalgebra of $Q(r)$.

Lemma 2. Let $i+j=n$. Then W is a subalgebra in the tensor product $W^i \otimes W^j$, where $W^r \subset U(\mathfrak{h}_r)$ denotes the W-algebra for $Q(r)$.

Corollary. If $i_1+\cdots+i_p=n$, then W is a subalgebra in $W^{i_1}\otimes\cdots\otimes W^{i_p}$.

Let $r, p, q \in \mathbb{N} \cup \{0\}$ and $r + 2p + q = n$,
$\mathbf{t} = (t_1, \ldots, t_p) \in \mathbb{C}^p$, $t_1, \ldots, t_p \neq 0$,
$\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathbb{C}^q$, $\lambda_1, \ldots, \lambda_q \neq 0$, such that $\lambda_i + \lambda_j \neq 0$ for any $1 \leq i \neq j \leq q$.

We have an embedding

$$W \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q.$$

Set

$$S(\mathbf{t}, \lambda) := \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p} \boxtimes V(\lambda).$$

Theorem 2. *(P.–S., 2019)*

(1) $S(\mathbf{t}, \lambda)$ is a simple W-module;

(2) Every simple W-module is isomorphic to $S(\mathbf{t}, \lambda)$ up to change of parity.

Idea of proof. Use Propositions (1)–(3).

Proposition 4.
Two simple modules $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda')$ are isomorphic if and only if $\mathbf{t}' = \sigma(\mathbf{t})$ and $\lambda' = \tau(\lambda)$ for some $\sigma \in S_p$ and $\tau \in S_q$.

8. Central characters

• The *center* Z of W coincides with the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism. It is generated by the *Q-symmetric polynomials (A. Sergeev)*:

$$p_k = x_1^{2k+1} + \cdots + x_n^{2k+1} \text{ for all } k \in \mathbb{N}$$

• Every $\mathbf{s} \in \mathbb{C}^n$ defines the central character $\chi_{\mathbf{s}} : Z \to \mathbb{C}$. Theorem 2–(2) implies that every simple W-module admits central character $\chi_{\mathbf{s}}$ for some $\mathbf{s}$.

Definition. For every $\mathbf{s} = (s_1, \dots, s_n)$ we define the *core* $c(\mathbf{s}) = (s_{i_1}, \dots, s_{i_m})$ as a subsequence obtained from $\mathbf{s}$ by removing all $s_j = 0$ and all pairs (s_i, s_j) such that $s_i + s_j = 0$. The core is well defined up to permutation.

Example. Let $\mathbf{s} = (1, 0, 3, -1, -1)$, then $c(\mathbf{s}) = (3, -1)$.

Lemma 3. Let $\mathbf{s}, \mathbf{s}' \in \mathbb{C}^n$. Then $\chi_{\mathbf{s}} = \chi_{\mathbf{s}'}$ if and only if $\mathbf{s}$ and $\mathbf{s}'$ have the same core (up to permutation) (A. Sergeev).
Thus the core depends only on the central character $\chi_{\mathbf{s}}$, we denote it $c(\chi)$.

• The category $W-\text{mod}$ of finite dimensional W-modules decomposes into direct sum $\bigoplus W^{\chi}-\text{mod}$, where $W^{\chi}-\text{mod}$ is the full subcategory of modules admitting generalized central character χ.

Lemma 4. A simple W-module S belongs to $W^{\chi}-\text{mod}$ if and only if it is isomorphic to $S(\mathbf{t},\lambda)$ with $\lambda=c(\chi)$.

Proof. We have to compute the central character of $S(\mathbf{t},\lambda)$. For a Q-symmetric polynomial $p_k=x_1^{2k+1}+\cdots+x_n^{2k+1}$ we have $p_k(\mathbf{t},\lambda)=\lambda_1^{2k+1}+\cdots+\lambda_q^{2k+1}$. Since p_k generate the center of W the statement follows.

We plan to describe blocks in the category of finite dimensional W-modules.

We did this for $Q(2)$.

9. Connection with super-Yangians

Super-Yangian $Y(Q(n))$ was introduced by M. Nazarov. *(Lecture Notes in Math. 1992)*

• $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)} \text{ where } m = 1, 2, \dots \text{ and } i, j = \pm 1, \pm 2, \dots, \pm n.$$

• The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j)$$

where $p(i) = 0$ if $i > 0$ and $p(i) = 1$ if $i < 0$.

• To write down defining relations for these generators we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{i,j} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots .$$

$$\begin{aligned}(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\ = (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\ - (u-v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}\end{aligned} \tag{1}$$

$$T_{i,j}(-u) = T_{-i,-j}(u) \tag{2}$$

• We proved that in the regular case the finite W-algebra for $Q(n)$ is a quotient of the super-Yangian of $Q(1)$ *(Adv. Math., 2016)*.

• We generalized this result for $Q(nl)$ when the corresponding nilpotent element has Jordan blocks each of size l. We proved that the finite W-algebra is a quotient of the super-Yangian of $Q(n)$ *(J. Math. Phys., 2017)*.

Our results should have applications to classification of simple modules for super-Yangians of type Q.

10. Whittaker coinvariants (work in progress)

- J. Brundan and S. Goodwin studied the *Whittaker coinvariants functor:* an exact functor from category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ to a certain category of finite-dimensional modules over $W_{m|n}$. (*Adv. Math. 2019*).

- Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, and $e \in \mathfrak{g}_{\bar{0}}$ be regular nilpotent.

Consider a good $\mathbb{Z}$-grading for e: $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$, $e \in \mathfrak{g}_1$. Let

$$\mathfrak{p} := \oplus_{i\geq 0}\mathfrak{g}_i, \ \mathfrak{h} := \mathfrak{g}_0, \ \mathfrak{m} := \oplus_{i<0}\mathfrak{g}_i.$$

Let $\chi \in \mathfrak{g}^*$ be given by $\chi(x) := (x|e)$.

Let I_χ be the *right* ideal of $U(\mathfrak{g})$ generated by $x - \chi(x)$ for all $x \in \mathfrak{m}$.

Define the *principal* W-algebra by

$$W := \{u \in U(\mathfrak{p}) \mid uI_\chi \subseteq I_\chi\}.$$

Let M be a *left* $\mathfrak{g}$-module. Then there is a well-defined *left* action of W on

$$H_0(M) := M/I_\chi M.$$

This gives the *Whittaker coinvariants functor*

$$H_0 : U(\mathfrak{g}) - \text{mod} \longrightarrow W - \text{mod}.$$

We plan to study the Whittaker coinvariants functor for $\mathfrak{g} = Q(n)$.

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