Regular and Graph Regular Operators on Hilbert C^*-Modules

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Dedicated to the memory of **LEONID VAKSMAN**

Contents

This talk is based on a joint work with
Rene Gebhardt: Internat. J. Math. **26**(2015).

Throughout this talk:

A denotes a C^*-algebra, E, F, G are Hilbert A-modules.

Pioneering work:

S. Baaj (Thesis 1980) and **S.L. Worononowicz** (1991, 1992).

Further work:
M. Hilsum (1987), Napiorkowski (1992), D. Kucerowsky (1997, 2002), A. Pal (1999), Pierrot (2006), M. Frank (2010), K. Sharifi (2010), J. Kaad/ M. Lesch (2012), R. Gebhardt/K.S. (2015), R. Meyer (2016) and many others.

Book "Hilbert C^*-modules" by C. Lance, London Math. Soc., Lecture Notes Series, 1995, treats regular operators.

Example: $A = C_0(\mathbb{R})$

All regular operators (according to Baaj, Woronowicz) are multiplication operators by continuous functions $f \in C(\mathbb{R})$. The multiplication operator by $f(x) = \frac{1}{x}$ is not regular, but it will be **graph regular**.

Aim of this talk:

To discuss new classes of operators on Hilbert C^*-modules: graph regular operators, orthogonally closed operators

Why operators on Hilbert C^*-modules?

Unbounded operators on Hilbert C^*-modules play an important role:

- **Noncommutative geometry** (Dirac operators), A. Connes
- **K-theory** (unbounded Fredholm modules), S. Baaj
- C^***-approach to quantum field theory** (observables)
- **Noncompact quantum groups** ("functions"), S.L. Woronowicz

Idea: $q \in \mathbb{C}, |q| = 1$. Algebraic quantum group

$$SL_q(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ab = qba,\ ac = qca,\ bc = cb,\right.$$
$$\left. ad - da = (q - q^{-1})bc \right\}.$$

Represent relations by Hilbert space operators and construct a C^*-algebra "generated" by these unbounded (!) operators.

For noncompact quantum groups unbounded operators on C^*-algebras are **crucial** and advanced results of the theory are really needed!

Definition of a Hilbert C^*-module

Definition: Hilbert C^*-module

A (right) **pre-Hilbert C^*-module** E over A is a right A-module E with a map $\langle\cdot,\cdot\rangle : E \times E \to A$ such that for $\alpha, \beta, \lambda \in \mathbb{C}, x, y, z \in E, a \in A$:

$$\langle \alpha x + \beta y, z\rangle = \overline{\alpha}\langle x, z\rangle + \overline{\beta}\langle y, z\rangle,$$
$$\langle x, ya\rangle = \langle x, y\rangle a,$$
$$\langle x, y\rangle = \langle y, x\rangle^*,$$
$$\langle x, x\rangle \geq 0 \quad \text{and} \quad \langle x, x\rangle = 0 \implies x = 0.$$

If $(E, \|.\|_E)$ is complete, where $\|.\|_E$ is the norm

$$\|x\|_E := \|\langle x, x\rangle\|_A^{1/2}, \quad x \in E,$$

then is a **Hilbert C^*-module** over A, briefly a **Hilbert A-module**.

Roughly speaking: $\langle\cdot,\cdot\rangle$ is an A-**valued scalar product on** E.
If $A = \mathbb{C}$, then a Hilbert A-module is just an "ordinary" Hilbert space.

Examples of Hilbert C^*-modules

Standard example: $E = A$

is a Hilbert A-module with multiplication as right action and

$$\langle a, b\rangle := a^* b, \quad a, b \in E.$$

Standard Hilbert C^*-module: $E = l_2(A)$

$$l_2(A) := \{(x_n)_{n=l}^{\infty} : x_n \in A, \ \sum_{n=1}^{\infty} x_n^* x_n \ \text{converges in} \ A\}$$

is a Hilbert A-module with pointwise operations and

$$\langle (x_n), (y_n)\rangle := \sum_{n=1}^{\infty} x_n^* y_n.$$

Caution: If A is infinite dimensional, then $l_2(A)$ is different from

$$H_1 := \{(x_n) : \sum_n \|x_n\|^2 < \infty\} \subset l_2(A).$$

The *orthogonal complement* of a subset G of E is the closed submodule

$$G^{\perp} := \{x \in E : \langle x, y\rangle = 0 \ \text{ for } \ y \in G\ \}.$$

Definitions: A submodule G of E is called

- **orthogonally closed** if $G = G^{\perp\perp}$,
- **orthogonally complemented** if $G \oplus G^{\perp} = E$,
- **essential** if $G^{\perp} = \{0\}$.

Each orthogonally closed submodule is closed.

Example: A proper closed submodule which is essential

Let $E = A = C([0, 1])$. Then $G = \{f \in E : f(0) = 0\}$ is a closed submodule of E such that $G \neq E$ and $G^{\perp} = \{0\}$.

Riesz' projection theorem does not hold for Hilbert C^*-modules!!!

Adjoint operators on Hilbert C^*-modules

Definition: An *operator* $t : E \to F$

is a $\mathbb{C}$-linear A-linear map defined on a right submodule $\mathcal{D}(t)$ of E:

$$t(\lambda x) = \lambda t(x) \quad \text{and} \quad t(xa) = t(x)a \quad \text{for} \quad \lambda \in \mathbb{C}, x \in \mathcal{D}(t), a \in A.$$

The A-linearity $t(xa) = t(x)a$ is a strong requirement!

Suppose $t : E \to F$ is **essentially defined**, that is, $\mathcal{D}(t)^\perp = \{0\}$. Set

$$\mathcal{D}(t^*) := \{y \in F | \exists\ z \in E : \langle tx, y\rangle_F = \langle x, z\rangle_E \ \text{ for } \ x \in \mathcal{D}(t)\ \}.$$

Since $\mathcal{D}(t)^\perp = \{0\}$, z is uniquely determined by y. Define $t^* y := z$. Then $t^* : F \to E$ is an operator, called the **adjoint** of t, and

$$\langle tx, y\rangle_F = \langle x, t^* y\rangle_E \quad \text{for} \quad x \in \mathcal{D}(t), y \in \mathcal{D}(t^*).$$

Definitions:

$t : E \to E$ is called **symmetric** if $t \subseteq t^*$, and **self-adjoint** if $t = t^*$.

Operators on Hilbert C^*-modules

Contrast to "ordinary" Hilbert space:

1. Self-adjoint operators are not necessarily densely defined!

2. Self-adjoint operators are not necessarily "good" operators.

(For instance, $(t + \mathrm{i})E$ and $(t^2 + 1)E$ are not dense in general!)

$\Rightarrow$ Further regularity conditions are needed.

Definitions:

Graph $\mathcal{G}(t) := \{(x, tx) : x \in \mathcal{D}(t)\}$ is a right A-submodule of $E \oplus F$.

An operator $t : E \to F$ is **orthogonally closed** if $\mathcal{G}(t)^{\perp\perp} = \mathcal{G}(t)$.

An orthogonally closed operator is closed, the converse does not hold.

Orthogonally closed operators behave "better" than closed operators.

The following definitions introduce the fundamental notions of the theory.

Definition: S. Baaj (1984), Woronowicz (1991)

A **closed** operator $t : E \to F$ is called **regular** if

$\mathcal{D}(t)$ is dense in E, $\mathcal{D}(t^*)$ is dense in F, and

$(1 + t^*t)E$ **is dense in** E. (Then $(1 + tt^*)F$ is also dense in F.)

$E = F = A$: regular operator = affiliated operator of Woronowicz (1991)

Definition: Gebhardt, K.S. (2015)

An **orthogonally closed** operator $t : E \to F$ is called **graph regular** if

$\mathcal{D}(t)^\perp = \{0\}$, $\mathcal{D}(t^*)^\perp = \{0\}$,

$(1 + t^*t)E$ **is dense in** E **and** $(1 + tt^*)F$ **is dense in** F.

$\mathrm{R}(E, F)$: regular operators, $\mathrm{R}_{gr}(E, F)$: graph regular operators.

"Regular operators are the most well-behaved operators."

Basic properties of graph regular operators

- Each regular operator is graph regular.
- A graph regular operator is regular if and only if it is densely defined.

Theorem 1: characterizations of graph regular operators

Let $t : E \to F$ be orthogonally closed. The following are equivalent:

(a) $t \in \mathrm{R}_{gr}(E, F)$, or equivalently, $t^* \in \mathrm{R}_{gr}(F, E)$.

(b) $\mathcal{G}(t)$ is orthogonally complemented: $\mathcal{G}(t) \oplus \mathcal{G}(t)^{\perp} = E \oplus F$.

(c) $(1 + t^*t)E = E$ and $(1 + tt^*)F = F$.

The assertions remain valid if we replace "orthogonally closed" by "closed" and "graph regular" by "regular".

Regularity criterion in terms of resolvent

Let $E = A$ be realized on a Hilbert space $\mathcal{H}$.

Multiplier algebra $M(A) = \{a \in \mathbf{B}(\mathcal{H}) : aA \subseteq A, Aa \subseteq A\}$.

Let T be a closed operator on $\mathcal{H}$ with non-empty resolvent set $\rho(T)$.

Theorem 2: K.S. (2004)

Let $\lambda \in \rho(T)$. Then T is regular on A if and only if $(T - \lambda I)^{-1} \in M(A)$, $(T - \lambda I)^{-1}A$ and $(T^* - \overline{\lambda} I)A$ are dense in A.

Example: $A = \mathcal{K}(\mathcal{H})$ compact operators, $M(A) = \mathbf{B}(\mathcal{H})$

Then T is **regular** iff T is a **densely defined closed** operator on $\mathcal{H}$.

"Ordinary" Hilbert space operator theory = operator theory on C^*-algebra of compact operators

Definition: An adjointable operator

is an operator of $\mathcal{L}(E,F) := \{t : E \to F : \mathcal{D}(t) = E, \mathcal{D}(t^*) = F\}$.

Adjointable operators are bounded, the converse is not true.

Each densely defined closed operator on a Hilbert space is a quotient of two bounded operators.

Examples: quotients "ba^{-1}" for $a \in \mathcal{L}(G,E)$, $b \in \mathcal{L}(G,F)$.

Suppose $\ker(a) \subseteq \ker(b)$, $\ker(a^*) = \{0\}$. Define

$$\mathcal{D}(t) = aG, \quad t(ax) = bx, x \in G.$$

If t is closed, then t is **graph regular** and $t^* = (a^*)^{-1}b^*$.

If $a \in \mathcal{L}(F,E)$ and $\ker(a) = \ker(a^*) = \{0\}$, then a^{-1} is **graph regular**.

Theorem 3: Napiorkowski/Woronowicz (1992)

Let G be a Lie group with Lie algebra g. Each $x \in g$ acts as left inv. diff. operator on G. Then x is a **regular operator** on the C^*-algebra $C^*(G)$.

Example: $E = A = C_0(\mathbb{R})$ continuous functions vanishing at infinity

Let $m : \mathbb{R} \to \mathbb{C}$. The multiplication operator t_m is defined by

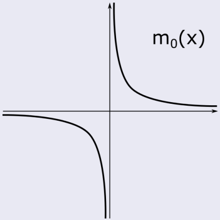
$$\mathcal{D}(t_m) := \{f \in C_0(\mathbb{R}) | m \cdot f \in C_0(\mathbb{R})\}, \quad t_m f := m \cdot f, f \in \mathcal{D}(t_m).$$

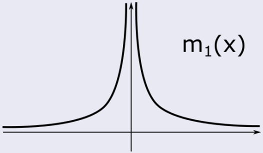
- Each **regular operator** is of the form t_m for some function $m \in C(\mathbb{R})$.

$$m(x) := \begin{cases} x^{-1}\exp(\mathrm{i}x^{-1}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}.$$

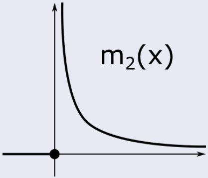
- t_m is **graph regular** and **normal** ($t_m^* t_m = t_m t_m^*$), but $\mathcal{D}(t_m) \neq \mathcal{D}(t_m^*)$.

Example: $E = A = C_0(\mathbb{R})$ continuous functions vanishing at infinity





- t_{m_0} and t_{m_1} are **graph regular**.
- t_{m_0} and t_{m_1} are **regular** for $E = A = C_0((-\infty, 0) \cup (0, +\infty))$.



- t_{m_2} is **not graph regular**.

Main tool: the bounded transform

Definition: Kaufman (1976)

Let T be a densely defined closed operator on a Hilbert space $\mathcal{H}$. The **bounded transform** of T is the operator

$$Z_T := T(I + T^*T)^{-1/2}.$$

Then $\|Z_T\| \leq 1$, $\ker\,(I - (Z_T)^* Z_T) = \ker(I + T^*T)^{-1} = \{0\}$,

$$T = Z_T(I - (Z_T)^* Z_T)^{-1/2}.$$

Conversely, if Z is a contraction such that $\ker\,(I - Z^*Z) = \{0\}$, then

$$T := Z(I - Z^*Z)^{-1/2}$$

is a densely defined closed operator T and $Z_T = Z$.

- $Z_{T^*} = (Z_T)^*$. Thus, Z_T is self-adjoint if and only if T is self-adjoint.
- Z_T is normal if and only if T is normal.

Graph regular operators and bounded transform

$\mathcal{Z}(E,F) := \{z \in \mathcal{L}(E,F) : \|z\| \le 1, \ker(I - z^*z) = \{0\}\}$

$\mathcal{Z}_r(E,F) := \{z \in \mathcal{L}(E,F) : \|z\| \le 1, (I - z^*z)E \text{ dense in } E\} \subseteq \mathcal{Z}(E,F)$.

For $z \in \mathcal{Z}(E,F)$ set $t_z := z(I - z^*z)^{-1/2}$.

(Note that $I - z^*z$ belongs to the C^*-algebra $\mathcal{L}(E,E)$.)

For $t \in \mathrm{R}_{gr}(E,F)$ the **bounded transform** of t is defined by

$$E_t := \overline{\mathcal{D}(t^*t)},\ F_{t^*} := \overline{\mathcal{D}(tt^*)},\ \ z_t := t(I + t^*t)^{-1/2} \restriction_{E_t} .$$

Theorem 4:

(a) $z \mapsto t_z$ is a bijection of $\mathcal{Z}_r(E,F)$ and $\mathrm{R}(E,F)$, $z = t_z(I + t_z^*t_z)^{-1/2}$.

(b) $z \mapsto t_z$ is an injection of $\mathcal{Z}(E,F)$ in $\mathrm{R}_{gr}(E,F)$.

(c) If $t \in \mathrm{R}_{gr}(E,F)$, then $t_{z_t} : E_t \to F_{t^*}$ is **regular**, called the **regular part** of the graph regular operator t.

The last theorem implies that $t(I-(z_t)^* z_t)^{1/2} = z_t$ if t is regular.

Theorem 5: Woronowicz (1991)

Let $A = E$ and $t \in \mathrm{R}(E)$. For **any** representation π of A there is a unique densely defined closed operator $\pi(t)$ such that $z_{\pi(t)} = \pi(z_t)$,

$$\pi(t)(\pi((I-(z_t)^* z_t)^{1/2} a)\varphi) = \pi(z_t a)\varphi, \quad a \in A, \varphi \in \mathcal{H}(\pi).$$

A regular operator is mapped to densely defined closed operator in all representations of A!

For instance:
Lie group representation $\Longrightarrow$ Lie algebra generators,
C^*-algebra of observables $\Longrightarrow$ unbounded observables

Graph regular operators can be "transported" only in **certain** (!) reps.

Operator theory for regular operators

There is an operator theory for **regular** operators on $E = A$.
It includes the following:

1. Polar decomposition
2. Functional calculus of normal operators
3. Stone's theorem
4. Self-adjoint extensions of symmetric operators via Cayley transform
5. Existence of self-adjoint extensions of positive symmetric operators
6. Kato-Rellich theorem for relatively bounded symmetric operators

1.-5.: Napiorkowski/Woronowicz (1992)- fundamental work on regular operators,
6. Gebhardt/K.S.

Lie algebra of the Heisenberg group

Heisenberg group H: group of matrices $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$.

The Lie algebra of H has a basis $\{X, Y, Z\}$ with commutation relations

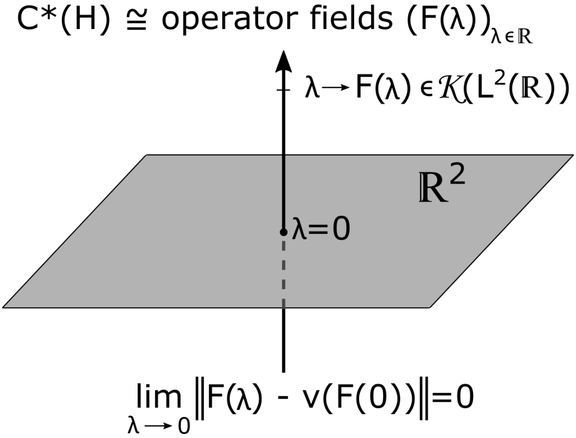
$$[X, Y] = Z, \ [X, Z] = [Y, Z] = 0.$$

Recall that Lie algebra elements acts as **regular** operators on $C^*(H)$.
Irreducible reps U of H are parametrized by $\lambda \in \mathbb{R}$, where $dU(Z) = \mathrm{i}\lambda I$:

$$\lambda = 0 : \ dU_a(X) = ia_1, dU_a(Y) = ia_2, dU_a(Z) = 0, \ a = (a_1, a_2) \in \mathbb{R}^2, \ \text{on } \mathbb{C}$$

$$\lambda \neq 0 : \ dU_\lambda(X) = -\mathrm{i}\lambda x, dU_\lambda(Y) = \frac{d}{dx}, dU_\lambda(Z) = \mathrm{i}\lambda I \ \text{ on } \ L^2(\mathbb{R}).$$

Group C^*-algebra $C^*(H)$: operator fields, Ludwig/Turowska (2011).



Here $F(0) \in \mathbf{B}(L^2(\mathbb{R}^2))$ and $v : \mathbf{B}(L^2(\mathbb{R}^2)) \to \mathbf{B}(L^2(\mathbb{R}))$.

Proposition:

Z^{-1} is a **graph regular** operator on the C^*-algebra $C^*(H)$.

Unbounded Toeplitz operators

Let $\phi \in L^\infty(\mathbb{T})$. T_ϕ is the Toeplitz operator: $T_\phi f := P\phi f, f \in H^2(\mathbb{T})$.
C^*-algebra generated by unilateral shift $S = T_z$ is the **Toeplitz algebra**

$$\mathcal{T} := \{T_\phi | \phi \in C(\mathbb{T})\} \dot{+} \mathcal{K}(H^2(\mathbb{T})).$$

Let $p, q \in \mathbb{C}[z]$ be relatively prime such that $q \neq 0$ in $\mathbb{D}$. Define

$$\mathcal{D}(T_{p/q}) := \{f \in H^2(\mathbb{T}) : (p/q)f \in H^2(\mathbb{T})\}, T_{p/q}f := (p/q)f, f \in \mathcal{D}(T_{p/q}).$$

Proposition:

$T_{p/q}$ is a **graph regular** operator for the Toeplitz algebra $E = \mathcal{T}$.
If q has a zero on $\mathbb{T}$, then $T_{p/q}$ is not regular.

For instance, $(S - I)^{-1}$ is graph regular, but not regular.

A fraction algebra (resolvent algebra in quantum physics)

Let $P = -\mathrm{i}\frac{d}{dx}$ and $Q = x$ be the self-adjoint operators on $L^2(\mathbb{R})$. Then

$$a := (Q - \mathrm{i}I)^{-1} \quad \text{and} \quad b := (P - \mathrm{i}I)^{-1}. \tag{1}$$

satisfy the commutation relations

$$a - a^* = 2\mathrm{i}a^*a = 2\mathrm{i}aa^*,\ b - b^* = 2\mathrm{i}b^*b = 2\mathrm{i}bb^*, \tag{2}$$

$$ab - ba = -\mathrm{i}ab^2a = -\mathrm{i}ba^2b,\ ab^* - b^*a = -\mathrm{i}a(b^*)^2a = -\mathrm{i}b^*a^2b^*. \tag{3}$$

Let A be the unital C^*-algebra generated by a and b satisfying (2) and (3). It occurs in the work of Buchholz, Grundling etc in quantum physics.

The irreducible reps of A are given by the operators (1) on $L^2(\mathbb{R})$ and one dimensional reps given by the points of the circles

$$K_1 := \{(a, 0) \in \mathbb{C}^2 : a - \overline{a} = 2\mathrm{i}|a|^2\}, K_2 := \{(0, b) \in \mathbb{C}^2 : b - \overline{b} = 2\mathrm{i}|b|^2\}.$$

Define operators q and p on the C^*-algebra A by

$$q := \mathrm{i}I + a^{-1}, \mathcal{D}(q) := aA \quad \text{and} \quad p := \mathrm{i} + b^{-1}, \mathcal{D}(p) := bA,$$

Proposition :

p and p are **graph regular** self-adjoint operators on $E := A$.

q and p are not regular operators on A, since their domains aA and bA are even not dense in $E = A$.

Their restrictions are **regular operators** for the essential ideal $\mathcal{K}(L^2(\mathbb{R}))$ of the C^*-algebra A. These restrictions are the regular parts of q and p.

Why graph regular operators?

Isolated singularities of holomorphic functions such as

$$f_1(z) = z^{-1}, f_2(z) = \exp(z^{-1})$$

are studied by the behavior in a neighborhood of the singularity.
Many C^*-algebras consist of operator fields $z \to a(z) \in A_z$, where A_z is a C^*-algebra on a Hilbert space $\mathcal{H}_z$.

Consider now an operator field $z \to t(z)$ such that $t(z), z \neq z_0$, is a **regular operator** for the C^*-algebra A_z. In general, $t(z_0)$ is not defined. Many graph regular operators are of this form.

Examples:

$t(x) = x^{-1}$ for $A = C_0(\mathbb{R})$,
Z^{-1} for the C^*-algebra $C^*(H)$,
$(S - I)^{-1}$ for the Toeplitz algebra.

Graph regular operators are a tool for the study of operator fields with isolated singulatities