On Noncommutative Fiber Bundles

Wojciech Szymański

University of Southern Denmark, Odense

Kristineberg, June 6, 2019

Classical compact (locally trivial) fiber bundles

$F \longrightarrow T \longrightarrow B$

compact topological spaces	$C(B) \subset C(T)$, $C(T) \to C(F)$
differentiable (polynomial) functions	$\mathcal{O}(B) \subset \mathcal{O}(T)$, $\mathcal{O}(F)$

Classical compact (locally trivial) fiber bundles

$F \longrightarrow T \longrightarrow B$

compact topological spaces $C(B) \subset C(T)$, $C(T) \to C(F)$
differentiable (polynomial) functions $\mathcal{O}(B) \subset \mathcal{O}(T)$, $\mathcal{O}(F)$

Principal bundles

F compact (Lie) group
$F \times T \longrightarrow T$ free action
$B = T/F$ orbit space

Classical compact (locally trivial) fiber bundles

$F \longrightarrow T \longrightarrow B$

compact topological spaces $C(B) \subset C(T)$, $C(T) \to C(F)$
differentiable (polynomial) functions $\mathcal{O}(B) \subset \mathcal{O}(T)$, $\mathcal{O}(F)$

Principal bundles

F compact (Lie) group
$F \times T \longrightarrow T$ free action
$B = T/F$ orbit space

Global aspects

structure of the C^*-algebras
homology/K-theory
connections
projectivity

Noncommutative bundles?

- NC principal bundles = fairly well understood
- general fiber bundles = ???
  $\Rightarrow$ ongoing project with T, Brzeziński and S. E. Mikkelsen
- example driven approach:

$$F_q \longrightarrow T_q \longrightarrow B_q$$

q-deformations of classical (sphere) bundles

Noncommutative principal bundles

- $A^H \subset A$
- H Hopf algebra (compact quantum group)
- $1 \in A$, C^*-algebra
- $\rho_A : A \to A \otimes H$ free coaction
  unital, $*$-algebra homomorphism
  $(\rho_A \otimes \mathrm{id})\rho_A = (\mathrm{id} \otimes \Delta_H)\rho_A$ (coassociative)
  $(\mathrm{id} \otimes \epsilon_H)\rho_A = \mathrm{id}$
- $A^H = \{a \in A \mid \rho_A(a) = a \otimes 1\}$ fixed point subalgebra
- $\mathcal{A} \subset A$ dense unital $*$-subalgebra, H-invariant

Example: Quantum Hopf fibration

Classical

$S^1 \longrightarrow S^3 \longrightarrow S^2$ or, equivalently, $U(1) \longrightarrow SU(2) \longrightarrow \mathbb{C}P^1$

Example: Quantum Hopf fibration

Classical

$S^1 \longrightarrow S^3 \longrightarrow S^2$ or, equivalently, $U(1) \longrightarrow SU(2) \longrightarrow \mathbb{C}P^1$

q-deformed

$U(1)$ acts on $C(SU_q(2))$ as subgroup of a quantum group

fixed point subalgebra $= C(\mathbb{C}P^1_q) =$ standard Podleś sphere

Example: Quantum Hopf fibration

Classical

$S^1 \longrightarrow S^3 \longrightarrow S^2$ or, equivalently, $U(1) \longrightarrow SU(2) \longrightarrow \mathbb{C}P^1$

q-deformed

$U(1)$ acts on $C(SU_q(2))$ as subgroup of a quantum group

fixed point subalgebra $= C(\mathbb{C}P^1_q) =$ standard Podleś sphere

Associated bundles

$\forall m \in \widehat{U(1)} = \mathbb{Z}$ there is an associated quantum line bundle with the Chern number m

$\mathcal{O}(SU_q(2)) =$ direct sum of these quantum line bundles

Noncommutative principal bundles cont.

Strong connection

$\rho_A : \mathcal{A} \to \mathcal{A} \otimes H$ right coaction

$\ell : H \to \mathcal{A} \otimes \mathcal{A}$ strong connection:

$$\mu_{\mathcal{A}} \circ \ell = 1_A \circ \epsilon_A$$

splitting of the multiplication $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
Also ℓ must be left- and right-colinear and normalized

Noncommutative principal bundles cont.

Strong connection

$\rho_A : \mathcal{A} \to \mathcal{A} \otimes H$ right coaction

$\ell : H \to \mathcal{A} \otimes \mathcal{A}$ strong connection:

$$\mu_{\mathcal{A}} \circ \ell = 1_A \circ \epsilon_A$$

splitting of the multiplication $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$
Also ℓ must be left- and right-colinear and normalized

Principal action

If $\exists \ell$ strong connection then

$$\mathrm{can} : \mathcal{A} \otimes_{\mathcal{A}^H} \mathcal{A} \longrightarrow \mathcal{A} \otimes H, \qquad a \otimes a' \mapsto (a \otimes 1)\rho_A(a')$$

is bijective

Associated projective modules

$\lambda_V : V \to H \otimes V$, left H-comodule
V vector space

$$\mathcal{A}\Box_H V := \{x \in \mathcal{A} \otimes V \mid (\rho_A \otimes \mathrm{id})(x) = (\mathrm{id} \otimes \lambda_V)(x)\}$$

cotensor product of $\mathcal{A}$ and V is

- projective $\mathcal{A}^H$-module
- finitely generated if V is finite dimensional

Connections $\Rightarrow$ modules

Associated projective modules

$\lambda_V : V \to H \otimes V$, left H-comodule
V vector space

$$\mathcal{A}\Box_H V := \{x \in \mathcal{A} \otimes V \mid (\rho_A \otimes \mathrm{id})(x) = (\mathrm{id} \otimes \lambda_V)(x)\}$$

cotensor product of $\mathcal{A}$ and V is

- projective $\mathcal{A}^H$-module
- finitely generated if V is finite dimensional

Explicit projectors

There is an explicit (but complicated) formula for writing down a projector corresponding to $\mathcal{A}\Box_H V$

Search for noncommutative sphere bundles

- Example driven search for the right concept
- Simultaneous C^*-algebraic and purely algebraic framework
- Sources of examples: q-deformations of weighted lens and projective spaces, flag manifolds, ...
- Problems:
  - The very concept of **fiber** in the noncommutative setting
  - A K-theoretic analogue of the classical **Gysin sequence**:

    $$\cdots \longrightarrow H^n(T) \longrightarrow H^{n-k}(B) \longrightarrow H^{n+1}(B) \longrightarrow H^{n+1}(T) \longrightarrow \cdots$$

    for a sphere bundle $S^k \longrightarrow T \longrightarrow B$
  - **Connection** in a general bundle?

Quantum flag manifold $SU_q(3)/\mathbb{T}^2$

Classical case

$$\mathbb{C}P^1 \longrightarrow SU(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P^2$$

The flag manifold $SU(3)/\mathbb{T}^2$ = the orbit space for the natural action of the maximal torus $\mathbb{T}^2$ on $SU(3)$

Quantum flag manifold $SU_q(3)/\mathbb{T}^2$

Classical case

$$\mathbb{C}P^1 \longrightarrow SU(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P^2$$

The flag manifold $SU(3)/\mathbb{T}^2$ = the orbit space for the natural action of the maximal torus $\mathbb{T}^2$ on $SU(3)$

q-deformation

- $SU_q(3)$ = quantum $SU(3)$ group of Woronowicz
- $\mathbb{T}^2$ = maximal torus (classical subgroup) in $SU_q(3)$
- $\mathbb{C}P^2_q$ = quantum complex projective 2-space of Vaksman and Soibelman
- $\mathbb{C}P^1_q$ = standard Podleś sphere
  = quantum complex projective 1-space

The C^*-algebra

base and fiber

- $C(\mathbb{C}P^2_q) := C(SU_q(3)/\mathbb{T}^2) \cap C^*(u_{11}, u_{21}, u_{31})$
- $\exists \pi : C(SU_q(3)/\mathbb{T}^2) \longrightarrow C(\mathbb{C}P^1_q)$ surjective $*$-homo. s.t.
  $\pi(C(\mathbb{C}P^2_q)) = \mathbb{C}1$
  $C(\mathbb{C}P^1_q)$ is the largest C^*-algebra with this property

$C(\mathbb{C}P^1_q)$ = minimal unitization of the compacts

The C^*-algebra

base and fiber

- $C(\mathbb{C}P^2_q) := C(SU_q(3)/\mathbb{T}^2) \cap C^*(u_{11}, u_{21}, u_{31})$
- $\exists \pi : C(SU_q(3)/\mathbb{T}^2) \longrightarrow C(\mathbb{C}P^1_q)$ surjective $*$-homo. s.t.
  $\pi(C(\mathbb{C}P^2_q)) = \mathbb{C}1$
  $C(\mathbb{C}P^1_q)$ is the largest C^*-algebra with this property

$C(SU_q(3)/\mathbb{T}^2)$	AF	6 irr. rep.	1 faithful	1 character
$C(\mathbb{C}P^2_q)$	AF	3 irr. rep.	1 faithful	1 character
$C(\mathbb{C}P^1_q)$	AF	2 irr. rep.	1 faithful	1 character

$C(\mathbb{C}P^1_q)$ = minimal unitization of the compacts

K-theory

$$
\begin{aligned}
K_0(C(SU_q(3)/\mathbb{T}^2)) &\cong \mathbb{Z}^6 \\
K_0(C(\mathbb{C}P_q^2)) &\cong \mathbb{Z}^3 \\
K_0(C(\mathbb{C}P_q^1)) &\cong \mathbb{Z}^2
\end{aligned}
$$

K-theory

$$\begin{aligned} K_0(C(SU_q(3)/\mathbb{T}^2)) &\cong \mathbb{Z}^6 \\ K_0(C(\mathbb{C}P_q^2)) &\cong \mathbb{Z}^3 \\ K_0(C(\mathbb{C}P_q^1)) &\cong \mathbb{Z}^2 \end{aligned}$$

$\exists$ a basis of $K_0(C(SU_q(3)/\mathbb{T}^2))$

$[P_1 = 1]_0$, $[P_2]_0$, $[P_3]_0$, $[Q_1]_0$, $[Q_2]_0$, $[Q_2]_0$, such that

P_j matrix projection over $\mathcal{O}(\mathbb{C}P_q^2)$

Q_j matrix projection over $\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ such that
$\pi_*([Q_j]_0 - [1]_0) = \pm$ non-trivial generator of $K_0(C(\mathbb{C}P_q^1))$

Polynomial algebra

Cotensor product decomposition

$U_q(2)$ acts on $\mathcal{O}(SU_q(3))$ from right
$U_q(2)$ acts on $\mathcal{O}(\mathbb{C}P^1_q)$ from left

$$\mathcal{O}(SU_q(3))\square_{U_q(2)}\mathcal{O}(\mathbb{C}P^1_q) \cong \mathcal{O}(SU_q(3)/\mathbb{T}^2)$$

as left $\mathcal{O}(\mathbb{C}P^2_q)$-modules

Polynomial algebra

Cotensor product decomposition

$U_q(2)$ acts on $\mathcal{O}(SU_q(3))$ from right
$U_q(2)$ acts on $\mathcal{O}(\mathbb{C}P^1_q)$ from left

$$\mathcal{O}(SU_q(3))\Box_{U_q(2)}\mathcal{O}(\mathbb{C}P^1_q) \cong \mathcal{O}(SU_q(3)/\mathbb{T}^2)$$

as left $\mathcal{O}(\mathbb{C}P^2_q)$-modules

Projectivity

$\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ is projective (inf. gen.) $\mathcal{O}(\mathbb{C}P^2_q)$-module

Polynomial algebra

Cotensor product decomposition

$U_q(2)$ acts on $\mathcal{O}(SU_q(3))$ from right
$U_q(2)$ acts on $\mathcal{O}(\mathbb{C}P^1_q)$ from left

$$\mathcal{O}(SU_q(3))\Box_{U_q(2)}\mathcal{O}(\mathbb{C}P^1_q) \cong \mathcal{O}(SU_q(3)/\mathbb{T}^2)$$

as left $\mathcal{O}(\mathbb{C}P^2_q)$-modules

Projectivity

$\mathcal{O}(SU_q(3)/\mathbb{T}^2)$ is projective (inf. gen.) $\mathcal{O}(\mathbb{C}P^2_q)$-module

Non-triviality

$C(SU_q(3)/\mathbb{T}^2)$ is not isomorphic to $C(\mathbb{C}P^2_q) \otimes C(\mathbb{C}P^1_q)$

- both are AF, same K_0-groups, same primitive ideals, but
- the hull-kernel topologies on the primitive ideal spaces are different!

Quantum twistor bundle

Classical

$$SU(2) \longrightarrow S^7 \longrightarrow S^4$$

instanton bundle (principal)
$\Rightarrow$ quotient out $U(1)$ (torus in $SU(2)$ $\Rightarrow$

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3 \longrightarrow S^4$$

Penrose twistor bundle

Quantum twistor bundle

Classical

$$SU(2) \longrightarrow S^7 \longrightarrow S^4$$

instanton bundle (principal)
$\Rightarrow$ quotient out $U(1)$ (torus in $SU(2)$ $\Rightarrow$

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3 \longrightarrow S^4$$

Penrose twistor bundle

q-deformation

several inequivalent deformations of the instanton bundle have been constructed with:

- $SU_q(2)$ as quantum structure group,
- various deformations of spheres S^7 and S^4

BCDT quantum instanton bundle

F. Bonechi, N. Ciccoli, L. Dąbrowski, M. Tarlini (2002, 2004)

$$SU_q(2) \longrightarrow S^7_q \longrightarrow S^4_q$$

noncommutative principal bundle

- S^7_q = Vaksman-Soibelman q-sphere
- $C(S^4_q)$ = minimal unitization of compacts
- $SU_q(2)$ coaction NOT algebra homomorphism !!!

Quantum twistor bundle cont.

for a certain choice of a free action of $U(1)$ on $C(S_q^7)$

$$\mathbb{C}P_q^1 \longrightarrow \mathbb{C}P_q^3 \longrightarrow S_q^4$$

still under investigation, but looks OK as a quantum sphere bundle

Quantum twistor bundle cont.

for a certain choice of a free action of $U(1)$ on $C(S_q^7)$

$$\mathbb{C}P_q^1 \longrightarrow \mathbb{C}P_q^3 \longrightarrow S_q^4$$

still under investigation, but looks OK as a quantum sphere bundle

- S_q^4 = the quantum 4-sphere of Bonechi, Ciccoli and Tarlini
- $\mathbb{C}P_q^3$ = the Vaksman-Soibelman quantum complex projective 3-space as C^*-algebra, but not as polynomial algebra

Thank you!