Geometry and Analysis on Kepler Manifolds

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A **hermitian Jordan triple** is a complex vector space $Z$, endowed with a ternary composition $Z \times Z \times Z \to Z$, denoted by

$$(x, y, z) \mapsto \{x; y; z\},$$

which is bilinear symmetric in $(x, z)$ and anti-linear in the inner variable, and satisfies the Jordan triple identity

$$[L(x, y), L(u, v)] = L(\{x; y; u\}, v) - L(u, \{v; x; y\})$$

where

$$L(x, y)z := \{x; y; z\}.$$

Moreover, the hermitian form

$$(x, y) \mapsto \text{trace } L(x, y)$$

is positive definite. Let $K$ denote the compact group of all linear transformations of $Z$ preserving the triple product

$$k\{u; v; w\} = \{ku; kv; kw\}.$$
Define the **quadratic representation**

\[ Q_z w := \frac{1}{2} \{ z; w; z \} \]

and the **Bergman endomorphisms**

\[ B(u, v) z = z - \{ u; v; z \} + \frac{1}{4} \{ u; \{ v; z; v \}; u \} \]

for \( u, v, z \in Z \). Thus

\[ B(u, v) = \text{id} - L(u, v) + Q_u Q_v \]

This is a complex-linear endomorphism of \( Z \), although \( Q_u \) and \( Q_v \) are conjugate-linear. Later we will use the **Jordan triple determinant**

\[ \Delta(u, v) = \det B(u, v)^{1/p} \]
Consider the matrix triple $Z = \mathbb{C}^{r \times s}$, with $r \leq s$.

\[ \{x; y; z\} = xy^*z + zy^*x \]

\[ Qzw = zw^*z \]

\[ B(u, v)z = z - uv^*z - zv^*u + u(vz^*)^*u = (1 - uv^*)z(1 - v^*u) \]

\[ B(u, v) = L_{1-uv^*} R_{1-v^*u}. \]

\[ \Delta(z, w) = \det (I_r - zw^*) = \det (I_s - w^*z) \]

$K = U(r) \times U(s) : z \mapsto u z v, \ u \in U(r), \ v \in U(s)$

$\text{rank} = r \leq s, \ a = 2 \text{ complex case}, \ b = s - r$

$r = 1, \ Z = \mathbb{C}^{1 \times d}, \ \{x; y; z\} = (x|y)z + (z|y)x$

$K = U(d) \text{ unitary group}$

$d = 1, \ Z = \mathbb{C}, \ \{x; y; z\} = 2x\overline{y}z$

$K = U(1) \text{ 1-torus}$
A real vector space $X$ is called a **Jordan algebra** iff $X$ has a non-associative product $x, y \mapsto x \circ y = y \circ x$, satisfying the Jordan algebra identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y).$$

The **anti-commutator product**

$$x \circ y = (xy + yx)/2$$

of self-adjoint operators on a Hilbert space satisfies the Jordan identity. Every euclidean Jordan algebra $X$ has a **symmetric cone**

$$\Omega = \{x^2 : x \in X \text{ invertible}\}.$$
By a fundamental result of Jordan/v. Neumann/Wigner (1934), every (euclidean) Jordan algebra have a **classification** as **self-adjoint matrices**

\[ X \approx \mathcal{H}_r(K) = \{(x_{ij}) \in K^{r \times r}, \; x_{ij}^* = x_{ji}\} \]

endowed with the anti-commutator product. Here \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \) (quaternions), or \( K = \mathbb{O} \) (octonions) if \( r \leq 3 \). For \( r = 2 \) we obtain formal \( 2 \times 2 \)-matrices **real spin factor**

\[ \mathcal{H}_2(K) = \left\{ \begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} : \; \alpha, \delta \in \mathbb{R}, \; b \in K := \mathbb{R}^{d-2} \right\}. \]

Thus \( X \) is characterized by the **rank** \( r \) and the **multiplicity**

\[ a = \dim_{\mathbb{R}} K \]

The symmetric cone \( \Omega \) corresponds to the positive definite matrices \( \mathcal{H}_r^+(K) \).
For a (euclidean) Jordan algebra $X$, the **complexification**

$$Z := X^\mathbb{C} = X \oplus iX$$

becomes a hermitian Jordan triple via

$$\{u; v; w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

The Jordan triples arising this way are called of **tube type**. For example, the matrix triple $\mathbb{C}^{r \times s}$ is of tube type if and only if $r = s$. For the spin factor of rank $r = 2$ we obtain $Z = \mathbb{C}^d$ with triple product

$$\{u; v; w\} = (u|v)w + (w|v)u - \overline{v}(u|\overline{w}).$$

$$K = T \cdot SO(n + 1)$$
Classification of hermitian Jordan triples:

- **Matrix triple** \( Z = \mathbb{C}^{r \times s} \) \( \text{rank} = r \leq s, \ a = 2 \) complex case, \( b = s - r \)
- **Symmetric matrices** \( a = 1 \) (real case)
- **Anti-symmetric matrices** \( a = 4 \) (quaternion case)
- **Spin factor** \( Z = \mathbb{C}^{n+1}, \ r = 2, \ a = n - 1, \ b = 0 \)
- **Exceptional Jordan triples** of dimension 16 \( (r = 2) \) and 27 \( (r = 3) \), \( a = 8 \) (octonion case), \( K = T \cdot E_6 \).

\[
d := \dim Z = r \left( 1 + \frac{a}{2}(r - 1) + b \right)
\]

\( r = \text{rank}(Z), \)  

\( a, b \) characteristic multiplicities

\( b = 0 \) tube type
For every hermitian Jordan triple $Z$ the **spectral unit ball**

$$D = \{ z \in Z : B(z, z) > 0 \} = \{ z \in Z : \| z \|_\infty < 1 \}$$

is a symmetric domain in its Harish-Chandra realization. This means that the group $G = \text{Aut}(D)$ of all biholomorphic automorphisms of $D$ acts transitively on $D$. The stabilizer subgroup $K$ at the origin $0 \in D$ consists of all linear transformations preserving the Jordan triple product. A basic theorem of M. Koecher asserts that, conversely, every symmetric domain can be realized as the unit ball of a unique hermitian Jordan triple $Z$. For matrices we obtain the matrix ball

$$D = \{ z \in \mathbb{C}^{r \times s} : I_r - zz^* > 0 \}.$$  

The domain $D$ (or the Jordan triple $Z$) is said to be of **tube type** if $D$ is biholomorphically equivalent to a tube domain over a symmetric cone $\Omega$. 

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The $G$-invariant **Bergman metric** on the tangent spaces $T_{z}D = Z$ is given by

$$(u|v)_{z} = (B(z, z)^{-1}u|v)_{0},$$

using the $K$-invariant inner product $(u|v)_{0}$ on $T_{0}D = Z$ given by

$$(u|v)_{0} := \text{tr} \ L(u, v).$$

For matrices, we have

$$(u|v)_{0} = \text{tr} \ L(u, v) = \text{trace} \ (L_{uv} + R_{v}u) = (r + s) \ \text{tr} \ uv^{*},$$

and the Bergman metric at $z \in D$ is

$$(u|v)_{z} = (r + s) \ \text{tr} \ (1 - zz^{*})^{-1} \ u(1 - z^{*}z)^{-1} \ v^{*}. $$
Boundary of symmetric domains
An element \( c \in Z \) is called a **tripotent** if

\[
\{ c; c; c \} = 2c
\]

or, equivalently, \( Q_c c = c \). For \( Z = C^{r \times s} \) tripotents satisfy \( c c^* c = c \) and are called partial isometries. Every tripotent \( c \) induces a **Peirce decomposition**

\[
Z = Z^c_2 \oplus Z^c_1 \oplus Z^c_0,
\]

where

\[
Z^c_j = \{ z \in Z : \{ c; c; z \} = 2jz \}
\]

is an eigenspace of \( L(c, c) \). Moreover, the Peirce 2-space \( Z^c_2 \) becomes a Jordan *-algebra with unit element \( c \), multiplication

\[
(z, w) \mapsto \frac{1}{2} \{ z; c; w \}
\]

and involution

\[
z \mapsto z^* = \frac{1}{2} \{ c; z; c \} = Q_c z
\]
The set of all tripotents is a (non-connected) real-analytic compact manifold, whose connected components

\[ S_\ell := \{ c \in Z \text{ tripotent} : \text{rank } c = \ell \}, \]

for \( 0 \leq \ell \leq r \), are \( K \)-homogeneous. Here the rank of a tripotent \( c \) is the rank of its Peirce 2-space \( Z_2^c \). We have \( S_0 = \{0\} \) and \( S_1 \) consists of all minimal (rank 1) tripotents. The maximal tripotents

\[ S := S_r = \partial_{ex} D \]

form the **Shilov (extreme) boundary** of \( D \).
One can show that the spectral norm satisfies

$$\|u + w\| = \max(\|u\|, \|w\|)$$

whenever $u \in Z_2^c$, $w \in Z_0^c$. Therefore, if $c \neq 0$, the set

$$c + D_0^c := \{c + w : w \in D \cap Z_0^c\}$$

belongs to the boundary $\partial D$, since $\|c\| = 1$. These are the so-called **boundary components** of $D$. They are pairwise disjoint and cover the whole boundary. The boundary components are precisely the (open) faces of $\partial D$ in the sense of convex geometry. Thus we have a disjoint union

$$\partial D = \bigcup (c + D_0^c).$$

over all non-zero tripotents $c$. The tripotent $c = 0$ gives the interior $D$. 
For any $1 \leq k \leq r$ the disjoint union

$$\partial_k D = \bigcup_{c \in S_k} (c + D_0^c)$$

is a $G$-orbit contained in the boundary. Thus

$$\partial D = \bigcup_{1 \leq k \leq r} \partial_k D$$

is a disjoint union of $G$-orbits, called the **boundary orbits**. If $k = r$ then

$$\partial_r D = S_r = S$$

is the **Shilov boundary** (realized as a $G$-orbit $S = G/P$) and the corresponding boundary components are the extreme points, since $Z_0^c = (0)$. For higher rank, the boundary is not smooth.
In the rank 1 case $Z = C^{1 \times d}$

$$S_1 = \{ u \in Z : (u|u) = 1 \} = S^{2d-1} = S$$

is the full boundary. For the spin factor $Z = C^d$ of $r = 2$, we have

$$\{ uv^* w \} = (u|v)w + (w|u)v - (u|w)v.$$ 

$$S = S_2 = T \cdot S^{d-1}$$

is called the **Lie sphere**, and the minimal tripotents

$$S_1 \approx S^*(S^n) = \{ (x, \xi) : \|x\| = 1, \|\xi\| = 1, (x|\xi) = 0 \}$$

form the **cosphere bundle** of $S^n$, which is a contact manifold.
Kepler varieties
A Jordan algebra $X$ has a **determinant function** $N(x)$ satisfying Cramer’s rule

$$x^{-1} = \frac{\text{grad}_x N}{N(x)}.$$

The Jordan algebra determinant $N : X \to \mathbb{R}$ is a **homogeneous** polynomial of degree $r$. For square matrices over $K = \mathbb{R}, \mathbb{C}$ we have

$$N(x) = \det(x).$$

For the rank 2 case,

$$N\begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} = \alpha\delta - (b|b)$$

is the Lorentz metric on $\mathbb{R}^{1,d-1}$. 

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For $\ell \leq r$ define the **Kepler variety**

$$Z_\ell = \{ z \in Z : \text{rank}(z) \leq \ell \}.$$  

Here $\text{rank}(z) \leq \ell$ if $N(z) = 0$ for all Jordan determinants of degree $> \ell$. Its regular part (**Kepler manifold**)  

$$\mathring{Z}_\ell = \{ z \in Z_\ell : \text{rank}(z) = \ell \}$$

is a $K^C$-homogeneous manifold, not compact or symmetric. The Kepler manifolds $\mathring{Z}_\ell$ are precisely the **Matsuki dual** $K^C$-orbits of the boundary $G$-orbits $\partial_\ell D$, with common intersection

$$\mathring{Z}_\ell \cap \partial_\ell D = S_\ell$$

being the $K$-orbit of tripotents.
For matrix spaces $Z = \mathbb{C}^{r \times s}$, $r \leq s$ and $1 \leq \ell \leq r$, we obtain the determinantal varieties

$$Z_{\ell} := \{ z \in Z : \text{rank}(z) \leq \ell \}$$

defined by vanishing of all $(\ell + 1) \times (\ell + 1)$-minors. The regular part is

$$\check{Z}_{\ell} = \{ z \in Z : \text{rank}(z) = \ell \} = Z_{\ell} \setminus Z_{\ell-1}.$$

In particular

$$Z_1 := \{ z \in Z : \text{rank}(z) \leq 1 \} \quad \text{vanishing of all } 2 \times 2 - \text{minors}$$

$$\check{Z}_1 := \{ z \in Z : \text{rank}(z) = 1 \} = Z_1 \setminus \{0\}$$

$$S_1 = \{ \xi \otimes \eta^* : \xi \in \mathbb{C}^r, \eta \in \mathbb{C}^s, \|\xi\| = 1 = \|\eta\| \}$$
For the spin factor $Z = \mathbb{C}^{n+1}$ of $r = 2$, the Jordan determinant

$$N(z) = \frac{1}{2}(z|\bar{z})$$

is the (complexified) Lorentz metric, and we recover the classical Kepler variety

$$Z_1 = \{ z \in \mathbb{C}^d : N(z) = 0 \} = \{ z = (z_0, \ldots, z_n) \in Z : z_0^2 + \ldots + z_n^2 = 0 \}$$

(complex light cone). Symplectic interpretation as cotangent bundle

$$\tilde{Z}_1 \approx T^*(S^n) \setminus \{ 0 \} = \{ (x, \xi) : \|x\| = 1, \xi \neq 0, (x|\xi) = 0 \}, \quad z = \frac{x + i\xi}{2}$$

$$n = \dim_{\mathbb{C}} Z_1 = p - 1 = d - 1$$

$$S_1 = \{ \frac{x + i\xi}{2} : |x| = |\xi| = 1, x \cdot \xi = 0 \} = S^*(S^{d-1})$$

cosphere bundle.

$$\dim S_1 = 2(d - 1) - 1 = 2n - 1$$
In joint work with M. Englis, we have studied reproducing kernels and their asymptotic expansion on Kepler manifolds.

**Theorem:** The Kepler variety $\mathcal{Z}_\ell$ is a normal variety having only rational singularities.

This result is classical for spin factors ($r = 2$) and proved by Kaup-Zaitsev for non-exceptional types. Our proof is uniform, based on Kempf’s collapsing vector bundle theorem.
Two tripotents \( u, v \in Z \) of the same rank \( \ell \) are said to be \textbf{Peirce equivalent} if

\[
Z^u_\lambda = Z^v_\lambda
\]

for all \( \lambda = 0, 1, 2 \). It is enough to consider the Peirce 2-space, since this determines the other two Peirce subspaces. The \( \ell \)-th \textbf{Peirce manifold} \( M_\ell \) associated with the Jordan triple \( Z \) is defined as the quotient space

\[
M_\ell := S_\ell / \sim
\]

of \( S_\ell \) under the Peirce equivalence relation. Thus \( M_\ell \) is the set of all Peirce 2-spaces \( U \subset Z \) having rank \( \ell \), in analogy with the classical Grassmann manifolds. It is known that \( M_\ell \) is a compact hermitian symmetric space for the semi-simple compact Lie group \( \dot{K} \) generated by the commutators in \( K \). Being a complex manifold, \( M_\ell \) is both a \( K \)-orbit and a \( K^C \)-orbit. The Peirce manifolds are the \textbf{Matsuki duals} of the open \( G \)-orbits (in general non-convex) in the conformal compactification \( \hat{Z} \).

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Clearly, $M_0$ consists only of $\{0\}$ and in the tube case $Z = X^C M_r = \{Z\}$, since $Z$ is the only Peirce 2-space of maximal rank. For non-tube domains, $M_r$ has higher dimension. For example, if $r = 1$, i.e., $Z = C^{1 \times d}$ viewed as row vectors, $M_1$ coincides with complex projective space $\mathbb{P}^{d-1}$. More generally, for $c \in S_\ell$ we may identify

$$M_\ell = \hat{Z}_1^C$$

with the compact dual space (conformal compactification) for the Peirce 1-space $Z_1^C$, viewed as a hermitian Jordan subtriple of $Z$. 
The first class of non-tube type Jordan triples are the rectangular matrix spaces

\[ Z = \mathbb{C}^{r \times (r+b)} \]

of rank \( r \), where \( b > 0 \). Choosing the maximal tripotent \( e = (1_r, 0) \) of rank \( r \), we obtain the Peirce 1-space

\[ Z_1^e = \{ (0, w) \mid w \in \mathbb{C}^{r \times b} \} \approx \mathbb{C}^{r \times b} \]

Its conformal hull gives the Grassmann manifold

\[ M_r = \hat{Z}_1^e = \text{Grass}_r(\mathbb{C}^{r+b}) \]

of all \( r \)-dimensional subspaces in \( \mathbb{C}^{r+b} \).
In the special case $r = 1$, corresponding to the unit ball in $Z = \mathbb{C}^{1+b}$, $e = (1, 0)$ is a unit vector and we have $Z^e_1 = \mathbb{C}^b$. Thus

$$M_1 = \text{Grass}_1(\mathbb{C}^{1+b}) = \mathbb{P}^b(\mathbb{C})$$

is the projective space. This is not the compact dual of the unit ball, which is

$$\hat{Z} = \text{Grass}_1(\mathbb{C}^{2+b}) = \mathbb{P}^{1+b}(\mathbb{C}).$$
For $U \in G_j(Z)$, the fibre

$$\pi_j^{-1}(U) = S(U) = S_j(Z) \cap U$$

coincides with the Shilov boundary of the unit ball $D \cap U$. One can show that different Peirce 1-spaces $U$ and $V$ of the same rank have disjoint Shilov boundaries:

$$U \neq V \implies S(U) \cap S(V) = \emptyset$$

We call the disjoint union

$$S_j(Z) = \bigcup_{U \in G_j(Z)} S(U),$$

the *tautological bundle* over $G_j(Z)$. Note that in Section 2, we realized $S_j(Z)$ as the base manifold of a bundle instead.
The Jordan triple automorphism group of $Z$ is given by

$$K = \text{Aut}(Z) = U(r) \times U(r + b)$$

acting on $Z$ by left and right multiplication $(a, d)(z) := azd^{-1}$ for $a \in U(r), \ d \in U(r + b)$ and $z \in \mathbb{C}^{r \times (r+b)}$. The complexification $K^\mathbb{C} = \text{GL}_r(\mathbb{C}) \times \text{GL}_{r+b}(\mathbb{C})$ acts in the same way. It follows that the action of $K$ on $G_r(Z)$ can be realized as the **collineation action**

$$K^\text{pr}_{2} \hookrightarrow U(r + b) \twoheadrightarrow \text{Aut}(\text{Grass}_r(\mathbb{C}^{r+b}))$$

of the second factor.
Define the **tautological vector bundle**

\[ T_\ell := \bigcup_{U \in M_\ell} U \to M_\ell. \]

Fix a base point \( c \in S_\ell \), put \( V := Z_c^2 \) and let \( L := \{ k \in K^C : kV = V \} \). Then \( T_\ell = K^C \times_L V \) becomes a **homogeneous vector bundle**, and

\[ T_\ell \to Z_\ell : U \ni z \mapsto z \in Z_\ell \]

is a collapsing map. By **Kempf’s Theorem**, the range of a collapsing map of a homogeneous vector bundle is a normal variety. Let \( \hat{U} \) denote the invertible elements of the Jordan algebra \( U = Z_2^u \). Then

\[ \bigcup_{U \in M_\ell} \hat{U} \to Z_\ell \]

is a birational isomorphism. This proves the normality theorem.
For any $\ell \leq r$, the **Kepler ball** is the intersection

$$\{z \in \hat{Z}_\ell : \|z\|_\infty < 1\} = \hat{Z}_\ell \cap D$$

of $\hat{Z}_\ell$ with the bounded symmetric domain $D := \{z \in Z : \|z\|_\infty < 1\}$. The compact $K$-homogeneous manifold $S_\ell$ of all rank $\ell$ tripotents can be regarded as the Shilov boundary of the Kepler ball $D_\ell$. It carries the normalized $K$-invariant measure

$$d\mu_\ell(u) = du.$$
Let $B(u, v)$ denote the Bergman operators on a hermitian Jordan triple $Z$.

**Proposition** For any tripotent $c \in S_\ell$, the holomorphic map

$$Z_1^c \to M_\ell, \quad v \mapsto [B(v, -c)c]$$

is a local chart of $M_\ell$. Here $[z]$ means the Peirce 2-space of an element $z \in \hat{Z}_\ell$ of rank $\ell$.

**Proposition** For any tripotent $c \in S_\ell$, the holomorphic map

$$Z_2^c \times Z_1^c \to T_\ell, \quad (u, v) \mapsto B(v, -c)u$$

becomes a vector-bundle chart of $T_\ell$. 
The above construction generalizes the well-known charts

$$\sigma_j(v_0, \ldots, \hat{v}_j, \ldots, v_d) := [v_0 : \ldots : 1 : \ldots : v_d] \in \mathbb{P}^d$$

for projective space, where $Z = \mathbb{C}^d$ and $0 \leq j \leq d$. In fact, putting $j = 0$ for simplicity, we choose $c = (1, 0, \ldots, 0) \in S_1$, with Peirce 1-space $Z_1^c = (0, \mathbb{C}^{d-1})$. Then for all $v \in \mathbb{C}^{d-1}$ we have

$$B(v, -c)c = (1 + (v|c))c(I_d + c^*v) = (1, 0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0, v) = (1, v).$$

Passing to homogeneous coordinates we obtain $[1 : v] = \sigma_0(v)$. 
The classical **blow-up process** used in algebraic geometry replaces a point in a complex manifold of dimension $n$ by a projective space $\mathbb{P}^{n-1}$. It is based on the incidence space

$$\mathcal{T} := \{(z, U) : z \in \mathbb{C}^n, U \in \mathbb{P}^{n-1}, z \in U\}$$

and the projection map

$$\pi : \mathcal{T} \to \mathbb{C}^n, \quad (z, U) \mapsto z,$$

whose fibre over $z \neq 0$ is the singleton $(z, [z])$, whereas the fibre over $0 \in \mathbb{C}^n$ is $\mathbb{P}^{n-1}$, regarded as a compact hermitian symmetric space of rank 1. For the Kepler varieties, we construct a blow-up process using compact hermitian symmetric spaces of higher rank.
Theorem The Kepler variety $Z_\ell$ of all elements of rank $\leq \ell$ has the regular (smooth part)

$$\check{Z}_\ell = \{ z \in Z : \text{rank}(x) = \ell \}.$$ 

Thus $Z_{\ell-1}$ is exactly the singular part of $Z_\ell$. Consider the incidence space

$$T_\ell = \{ (z, U) \in Z_\ell \times M_\ell : z \in U \}$$

endowed with the local charts constructed via the Bergman operators. Then the projection

$$\pi : T_\ell \rightarrow Z_\ell, \quad (z, U) \mapsto z,$$

is a proper holomorphic modification in the sense that for $z \in \check{Z}_\ell$ the fibre of $\pi$ is the singleton $(z, Z_z^\check{z})$, whereas for $z \in Z_{\ell-1}$ the fibres have higher dimension, in particular, the fibre over 0 is the full Peirce manifold $M_\ell$, which is a compact hermitian space of rank $\ell$. 
For a hermitian Jordan triple $Z$, consider the algebra $\mathcal{P}(Z)$ of all polynomials $p : Z \to \mathbb{C}$ and the natural action

$$(k^{-1} p)(z) := p(kz)$$

of $K = \text{Aut}(Z)$. Then there is a multiplicity-free Hua-Schmid-Kostant decomposition

$$\mathcal{P}(Z) = \sum_{m \in \mathbb{N}_+^r} \mathcal{P}_m(Z)$$

into pairwise inequivalent irreducible $K$-modules $\mathcal{P}_m(Z)$, where

$$m = m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$$

ranges over all integer partitions (Young diagrams). For $r = 1$ (Hilbert ball) $\mathcal{P}_m(Z)$ consists of all $m$-homogeneous polynomials.
The highest weight vector \( N_m \in \mathcal{P}_m(Z) \) can be described in terms of Jordan theoretic minors. Let \( e_1, \ldots, e_r \) be a frame of minimal tripotents of \( Z \), with joint Peirce decomposition

\[
Z = \sum_{0 \leq i \leq j \leq r} Z_{ij}.
\]

Then \( \mathcal{P}_m(Z) \) has the highest weight vector

\[
N_m(z) = N_1(z)^{m_1-m_2} N_2(z)^{m_2-m_3} N_r(z)^{m_r},
\]

where \( N_1, \ldots, N_r \) are the Jordan theoretic minors. For the simplest partition \( m = (1,0,\ldots,0) \), \( \mathcal{P}_m(Z) \) is the dual space of linear forms on \( Z \). In this case \( N_m(z) = (e_1|z) \).
In the trivial case, where $\ell = r$ is maximal and $Z$ is a Jordan algebra, negative powers of the (Jordan) determinant have no holomorphic extension beyond $\hat{Z}$. Apart from this we have

**Extension Theorem:** Every holomorphic function on $\hat{Z}_\ell$ has a unique extension to the closure $Z_\ell$

Proof: Singular set

$$Z_\ell \setminus \hat{Z}_\ell = \bigcup_{j < \ell} Z_j$$

has codimension $\geq 2$. By the Normality Theorem (and using Lojasiewicz), the **second Riemann extension theorem** holds for holomorphic functions on $\hat{Z}_\ell$. As a consequence, all $K$-invariant Hilbert spaces of holomorphic functions on $Z_\ell$ are completions of

$$\mathcal{P}^\ell(Z) = \sum_{m_1, \ldots, m_\ell, 0, \ldots, 0} \mathcal{P}_m(Z)$$

involving only partitions of length $\leq \ell$. 

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The classical Fock (or Segal-Bargmann) space $H^2(Z)$ of entire functions on $Z = \mathbb{C}^d$ has the inner product

$$(\phi|\psi)_Z := \frac{1}{\pi^d} \int_Z dz \ e^{-(z|z)} \overline{\phi(z)} \psi(z).$$

The polynomials $\mathcal{P}(Z)$ are dense in $H^2(Z)$ and we have

$$(p|q)_Z = (\partial_p q)(0)$$

where $\partial_p$ is the constant-coefficient differential operator associated with $p$ via the normalized inner product. The multi-variable Pochhammer symbol is

$$(\nu)_m = \prod_{j=1}^r (\nu - \frac{a}{2}(j - 1))_{m_j} = \frac{\Gamma_{\Omega}(\nu + m)}{\Gamma_{\Omega}(\nu)}$$

where $\Gamma_{\Omega}$ denotes the Gindikin Gamma-function.
Let $\Sigma \subset Z$ be a $K$-invariant set of uniqueness for holomorphic functions. A $K$-invariant measure $\mu$ with support $\Sigma$ is called **hypergeometric** of type $(x_0, \ldots, x_h)$, if for all partitions $m$ and all $\phi, \psi \in P_m(Z)$ we have

$$(\phi|\psi)_\mu := \int \mu(dz) \frac{\phi(z)}{\bar{\phi}(z)} \psi(z) = \frac{(y_1)_m \cdots (y_h)_m}{(x_0)_m \cdots (x_h)_m} (\phi|\psi)_Z.$$ 

The measure is uniquely determined by this property, since the spaces $P_m(Z)$ are mutually orthogonal.

If $\Sigma \subset Z_\ell$ and the identity holds only for partitions $m$ of length $\leq \ell$, we say that $\mu$ is $\ell$-hypergeometric.

Finding a hypergeometric measure $\mu$ of a given type amounts to solving a **moment problem** involving all partitions $m$. 
For each partition \( m \) choose an orthonormal basis \( p_\alpha \in \mathcal{P}_m(Z) \) and define the **Fischer-Fock reproducing kernel**

\[
E^m(z, w) = \sum_\alpha p_\alpha(z) p_\alpha(w).
\]

In the rank 1 case \( Z = C^{1 \times d} \) we have

\[
E^m(z, w) = \frac{(z \mid w)^m}{m!}.
\]

The condition above is equivalent to saying that the Hilbert space \( H^2_\mu(\Sigma) \) of all square-integrable holomorphic functions determined by \( \mu \) has the **reproducing kernel**

\[
q \mathcal{F}_p \left( \begin{array}{c} x_0, \ldots, x_h \\ y_1, \ldots, y_h \end{array} \right)(z, w) := \sum_m \frac{(x_0)^m \cdots (x_h)^m}{(y_1)^m \cdots (y_h)^m} E^m(z, w),
\]

given by a **hypergeometric series**. For \( z, w \in Z_\ell \), only partitions of length \( \leq \ell \) occur.
The Gauss measure

\[ \mu(dz) = dz \ e^{-(z|z)} \]

on \( \Sigma = \mathbb{Z} \) is hypergeometric of type \((\nu)\), since the Fock space (or Segal-Bargmann space) \( H^2_\mu(\mathbb{Z}) \) of entire functions on \( \mathbb{Z} \) has the \( 0\mathcal{F}_0 \)-type reproducing kernel

\[ e^{(z|w)} = \sum_m E^m(z,w). \]

Let \( \Delta \) be the Jordan triple determinant and \( \nu > p - 1 \). The measure

\[ \mu(dz) = dz \ \Delta(z,z)^{\nu-p} \]

on \( \Sigma = D \) (bounded symmetric domain) is hypergeometric of type \((\nu)\), since the weighted Bergman spaces \( H^2_\mu(D) \) has the reproducing kernel

\[ \Delta(z,w)^{-\nu} = \sum_m (\nu)_m \ E^m(z,w) \]

of type \( 1\mathcal{F}_0 \). This follows from the well-known Faraut-Korányi formula.
Let $\Sigma = \partial_k D$ be the $k$-th boundary orbit ($1 \leq k \leq r$). In joint work with J. Arazy, a measure $\mu$ on $\Sigma$ was defined in terms of polar decomposition, which is hypergeometric of type $\left( \nu_k \right)$, where

$$\nu_k = \frac{d}{r} + \frac{a}{2}(r - k)$$

are the so-called embedded Wallach parameters.

The case $\nu_r = \frac{d}{r}$ corresponds to the Hardy space, where $\Sigma = S = \partial_r D$ is the Shilov (extreme) boundary and $\mu$ is the $K$-invariant probability measure.

The case $\nu_1 = p - 1$, the left endpoint of the discrete series of weighted Bergman spaces, corresponds to the open boundary orbit $\partial_1 D$. 
For the Kepler manifold $\ell \leq r$ we obtain (joint with M. Englis)

- Let $\Sigma = \hat{Z}_\ell$ be the Kepler manifold, endowed with the $K$-invariant measure $\lambda_\ell$ induced by the Riemannian structure. Then the measure

$$\mu(dz) = e^{-(z|z)} \lambda_\ell(dz)$$

on $\Sigma$ is $\ell$-hypergeometric of type $(d_\ell/n_\ell)$, where $n = \dim Z_\ell$ and $d_\ell = \dim Z^c_2$ for $c \in S_\ell$. Thus the Fock-type space $H^2_{\mu}(Z_\ell)$ has a reproducing kernel of type $1F_1$.

- Let $\Sigma = D \cap Z_\ell$ be the Kepler ball. Then the (restricted) measure

$$\mu(dz) = \Delta(z, z)^{\nu-p} d\lambda_\ell(z)$$

is $\ell$-hypergeometric of $3F_2$-type $(r \frac{\alpha}{2}, \nu, \frac{\hat{d}}{\ell}, \frac{\hat{v}}{\nu_\ell})$.
Finally, one may combine Kepler manifolds and boundary orbits (joint with G. Misra)

- The boundary orbit $\Sigma = \partial_k D \cap Z_\ell$ of the Kepler ball ($k \leq \ell$) supports a $K$-invariant measure $\mu$ which is $\ell$-hypergeometric of type $(r^{\frac{a}{2}}, \nu_k, \frac{d}{r^\ell})$.

- For integration over the **partial Shilov boundary** $\Sigma = S_\ell$ of $D \cap Z_\ell$ ($k = \ell$) we obtain $(r^{\frac{a}{2}}, \frac{d}{r^\ell})$
Fock space on $Z = \mathbb{C}^d$, $\mathcal{K}_\nu = e^{(z|w)} = 0\mathcal{F}_0(z, w)$

Bergman space on $D$, $\mathcal{K}_\nu = \Delta(z, w)^{-\nu} = 1\mathcal{F}_0(z, w)$

Faraut-Koranyi formula

Kepler manifold $\tilde{Z}_\ell$, $\mathcal{K}_\nu = D_\ell 1\mathcal{F}_1$

Kepler ball $D \cap \tilde{Z}_\ell$, $\mathcal{K}_\nu = D_\ell 2\mathcal{F}_1$
For a given measure $\mu$ on $\Sigma$, consider the Hilbert space $H^2_\mu(\Sigma)$ of all square-integrable holomorphic functions, restricted to $\Sigma$, and denote by $P_\mu : L^2_\mu(\Sigma) \to H^2_\mu(\Sigma)$ the orthogonal projection. For continuous symbol function $f \in C(\Sigma)$ define the **Toeplitz operator**

$$T_f(\phi) := P_\mu(f \phi)$$

for all $\phi \in H^2_\mu(\Sigma)$. These operators are highly non-commutative and not essentially normal if the rank $r > 1$. Let

$$\mathcal{T}_\mu(\Sigma) = C^*(T_f : f \in C(\Sigma))$$

denote the **Toeplitz $C^*$-algebra**
Let $\mu$ be a $\ell$-hypergeometric measure on $\Sigma \subset \overline{D} \cap V_\ell$ (closed Kepler ball), of type $\left(\frac{x_0}{2}, \ldots, \frac{x_h}{2}, \frac{y_1}{2}, \ldots, \frac{y_h}{2}\right)$. Consider the stratification

$$\Sigma = \bigcup_{0 \leq i \leq \ell} \bigcup_{c \in S_i} c + \Sigma^c, \quad \Sigma^c := \Sigma \cap Z_0^c.$$

For $c \in S_\ell$ define a $H_c(z) := \exp(z|c)$. We regard $H^n_c$ as a peaking function on $c + \Sigma^c$.

**Theorem**

*For any tripotent $c \in S_1$ the sequence of probability measures*

$$\mu(dz) \frac{|H_c(z)|^{2n}}{\|H^n_c\|_\mu^2}$$

*converges to a measure $\mu^c$ supported on the boundary component $\Sigma^c$ which is $(\ell - 1)$-hypergeometric of type $\left(\frac{x_0}{2} - \frac{a}{2}, \ldots, \frac{x_h}{2} - \frac{a}{2}, \frac{y_1}{2} - \frac{a}{2}, \ldots, \frac{y_h}{2} - \frac{a}{2}\right)$.*
Let $\mathcal{T}_\mu(\Sigma)$ be the Toeplitz $C^*$-algebra on $H^2_\mu(\Sigma)$, associated with an $\ell$-hypergeometric measure supported on $\Sigma \subset \overline{D} \cap V_\ell$. Then for every $c \in S_i$, for $0 \leq i \leq \ell$, there is an irreducible $C^*$-representation

$$\sigma^c : \mathcal{T}_\mu(\Sigma) \to \mathcal{T}_\mu^c(\Sigma^c)$$

acting on $H^2_{\mu^c}(\Sigma^c)$, which is uniquely determined by the property

$$\sigma^c(T_\mu(f)) = T_\mu^c(f^c)$$

for all $f \in \mathcal{C}(\Sigma)$, where $f^c \in \mathcal{C}(\Sigma^c)$ is defined by

$$f^c(\zeta) := f(c + \zeta).$$

These representations are pairwise inequivalent and constitute the full spectrum of $\mathcal{T}_\mu(\Sigma)$. 
For the Hardy space $H^2(S)$, we obtain representations on the 'little' Hardy spaces $H^2(S^c)$ on the Shilov boundaries $S^c_0$ of rank $r - i$.

For the weighted Bergman spaces $H^2_{\nu}(D)$, $\nu > p - 1$, we obtain representations on the 'little' weighted Bergman spaces $H^2_{\nu - i\alpha/2}(D^c_0)$ on the boundary components $D^c_0$ of rank $r - i$.

For the boundary orbit spaces $H^2_{\nu_k}(\partial_k D)$, $k \leq \ell$ we obtain representations on the 'little' boundary orbit spaces if $i \leq k$ and 'little' weighted Bergman spaces if $k < i \leq \ell$. Thus some boundary orbits 'disappear' in the discrete series when passing to the boundary.
Let \((M, \omega)\) be a compact Kähler \(n\)-manifold, with a quantizing line bundle \(\mathcal{L} \rightarrow M\). Consider holomorphic sections \(\Gamma(M, \mathcal{L}^\nu)\) of a (high) tensor power \(\mathcal{L}^\nu\). **Kodaira (coherent state) embedding**

\[
\kappa_\nu : M \rightarrow \mathbf{P}(\Gamma(M, \mathcal{L}^\nu)), \quad z \mapsto [s_\nu^0(z), \ldots, s_\nu^N(z)]
\]

\[
D_\nu = \frac{i}{2} \partial \overline{\partial} \log \sum_{i=0}^{\nu N} |\zeta_i|^2 \text{ Fubini-Study metric on } \mathbf{P}^N
\]

\[
\kappa_\nu^*(D_\nu) = \nu \omega + \frac{i}{2} \partial \overline{\partial} \log T_\nu
\]

\[
T_\nu(z) = \sum_{i=0}^{d_\nu} h_z(s_\nu^i(z), s_\nu^i(z)) \text{ Kempf distortion function}
\]

**TYZ (Tian-Yau-Zelditch) expansion**

\[
\left| \frac{T_\nu(z)}{\nu^n} - \sum_{j=0}^{N-1} \frac{a_j(z)}{\nu^j} \right| \leq \frac{c_N}{\nu^N} \quad \text{as } \nu \rightarrow \infty
\]
**Theorem:** Consider the Fock-type situation, $\alpha = 1$, $\ell$ arbitrary. There exist polynomials $p_j(x)$, with $p_0 = \text{const}$, such that

$$
\frac{1}{\nu^n} K_\nu(\sqrt{x}, \sqrt{x}) \approx \frac{e^{\nu \text{tr}(x)}}{N_c(x)^n/\ell} \sum_{j=0}^{n} \frac{p_j(x)}{\nu^j},
$$

as $|x| \to +\infty$, uniformly for $x$ in compact subsets of $D_c$.

**Proof:** For $\text{Re}(\mu) > \frac{d}{\ell} - 1$ and $\text{Re}(\gamma) > \frac{d}{\ell} - 1$, there is an asymptotic expansion

$$
_1F_1\left(\begin{array}{c}
\mu \\
\gamma
\end{array}\right)(z) \approx \frac{\Gamma_r(\gamma)}{\Gamma_r(\mu)} \frac{e^{\text{tr}(z)}}{N(z)^{\gamma-\mu}} \, _2F_0\left(\begin{array}{c}
\frac{d}{\ell} - \mu; \gamma - \mu
\end{array}\right)(z^{-1})
$$

as $|z| \to +\infty$ while $\frac{z}{|z|}$ stays in a compact subset of $D_c$. This expansion can be differentiated termwise any number of times. Similar expansion for Kepler ball, based on asymptotics of $_2F_1$. 

Harald Upmeier  
Geometry and Analysis on Kepler Manifolds
Let $Z$ be an irreducible Jordan triple of rank $r$. For $1 \leq i < j \leq r$ put

$$a := \dim Z_{ij}, \quad b := \dim Z_{0j}$$

These **characteristic multiplicities** are determined by

$$\frac{d}{r} = 1 + \frac{a}{2}(r - 1) + b,$$

$$p = 2 + a(r - 1) + b,$$

where $p$ is the genus. In the matrix case $Z = \mathbb{C}^{r \times s}$ we have $a = 2, \ b = s - r$, and hence $d/r = s, \ p = r + s$. The **tube type** case corresponds to $b = 0$ or, equivalently, to $\frac{p}{2} = \frac{d}{r}$. Let

$$(u | v) = \frac{1}{p} \ tr_Z \ L(u, v)$$

be the normalized $K$-invariant inner product.
Segal-Bargmann-Fock space

Consider the **Fischer-Fock inner product**

\[
(\phi | \psi)_Z := \frac{1}{\pi^d} \int_Z d\pi \; e^{-(z|z)} \; \overline{\phi(z)} \; \psi(z) = (\partial \phi, \psi)(0)
\]

for all \( \phi, \psi \in \mathcal{P}(Z) \). Here \( \partial \phi \) denotes the constant coefficient differential operator on \( Z \) induced by \( \phi \) via the inner product \( (z|w) \). The Hilbert space completion of \( \mathcal{P}(Z) \) is the **Segal-Bargmann-Fock space**

\[
H^2(Z) = \{ \psi : Z \to \mathbb{C} \text{ holomorphic} : \int_Z \frac{d\pi}{\pi^d} \; e^{-(z|z)} |\psi(z)|^2 < \infty \}
\]

of entire functions, which has the reproducing kernel

\[
E(z, w) = e^{(z|w)}.
\]
$(M, \omega)$ compact Kähler $n$-manifold

$L \to M$ quantizing line bundle

$\mathcal{O}(M, L^\nu)$ holomorphic sections of power $L^\nu$

$\sigma_\nu : U \to L^\nu \setminus 0$ local trivializing section

$\nu \omega(z) = -\frac{i}{2} \overline{\partial} \partial \log h_\nu(\sigma_\nu(z), \sigma_\nu(z))$ Ricci form

$P(\mathcal{O}(M, L^\nu)), (s_0^\nu, \ldots, s_d^\nu)$ ONB of sections
\( \mathbb{P}^{d_{\nu}} : [Z^0, \ldots, Z^{d_{\nu}}] \) homogeneous coordinates

\( \kappa_{\nu} : M \ni z \mapsto [s^0_{\nu}(z), \ldots, s^{d_{\nu}}_{\nu}(z)] \in \mathbb{P}(\mathcal{O}(M, \mathcal{L}^\nu)) \) coherent state embedding

\[
D_{\nu} = \frac{i}{2} \partial \overline{\partial} \log \sum_{j=0}^{d_{\nu}} |Z^j|^2 \quad \text{Fubini-Study metric on } \mathbb{P}^{d_{\nu}}
\]

\[
\kappa_{\nu}^*(D_{\nu}) = \nu \omega + \frac{i}{2} \partial \overline{\partial} \log T_{\nu} = \frac{i}{2} \partial \overline{\partial} \log \frac{T_{\nu}}{h_{\nu}}
\]

\[
T_{\nu}(z) = \sum_{i=0}^{d_{\nu}} h_z(s^i_{\nu}(z), s^i_{\nu}(z)) \quad \text{distortion function}
\]

TYZ (Tian-Yau-Zelditch) expansion

\[
\frac{T_{\nu}(z)}{\nu^n} \approx \sum_{j \geq 0} \frac{a_j(z)}{\nu^j} \quad \text{as } \nu \to \infty
\]

\[
\left| \frac{T_{\nu}(z)}{\nu^n} - \sum_{j=0}^{N-1} \frac{a_j(z)}{\nu^j} \right| \leq \frac{c_N}{\nu^N}
\]
The 'partial' Hardy space, denoted by \( \mathcal{H}_{S_\ell} \), consists of all holomorphic functions on \( D_\ell \) having boundary values on \( S_\ell \) which are square-integrable for \( \mu_\ell \). The inner product is

\[
(\phi|\psi)_{S_\ell} := \int_{S_\ell} d\mu_\ell(u) \overline{\phi(u)} \psi(u).
\]

Passing to a Bergman type setting, consider the Kepler ball endowed with the probability measure

\[
d\mu_\ell(z) = d\mu_{\ell,\nu}(z) := c_\nu \Delta(z,z)^{\nu-p} d\lambda_\ell(z).
\]

Here \( d\lambda_\ell(z) \) is the 'Riemannian' measure on \( D_\ell \) induced by the normalized \( K \)-invariant hermitian metric \((z|w)\). The dependence on the spectral parameter \( \nu \) is tacitly understood. The 'partial' weighted Bergman space \( \mathcal{H}_{D_\ell} \) is defined as the Hilbert space of all holomorphic functions on the Kepler ball \( D_\ell \) which are square-integrable for \( \mu_\ell \). The inner product is

\[
(\phi|\psi)_{D_\ell} := \int_{D_\ell} d\mu_\ell(z) \overline{\phi(z)} \psi(z).
\]
Since the underlying measure is $K$-invariant, the second Riemann extension theorem (a consequence of the normality theorem) implies that a Hilbert space $\mathcal{H}$ of holomorphic functions on the Kepler variety is a Hilbert sum

$$\mathcal{H} = \sum \mathcal{P}_{m_1, m_\ell, 0, \ldots, 0}(Z)$$

involving only (non-negative) partitions of length $\leq \ell$, since all other components vanish on $Z_\ell$.

**Proposition**: Let $m \in \mathbb{N}_\ell^\ell$. Then we have for all $p_m, q_m \in \mathcal{P}_Z^m$

$$ (p_m | q_m)_{S_\ell} = \frac{(\frac{a}{2} \ell)^m}{(\frac{a}{2} r)^m} \frac{(p_m | q_m)}{(1 + \frac{a}{2}(r - 1) + b)^m}, $$

$$ (p_m | q_m)_{D_\ell} = \frac{(\frac{a}{2} \ell)^m}{(\frac{a}{2} r)^m} \frac{(1 + \frac{a}{2} (2r - \ell - 1) + b)^m}{(1 + \frac{a}{r} - 1) + b)^m} \frac{(p_m | q_m)}{(\nu)^m} $$
Choose a base point $c = e_1 + \ldots + e_\ell \in S_\ell$. Every positive density function $\rho(t)$ on symmetric cone $\Omega_c$ induces a $K$-invariant radial measure

$$\int_{\mathcal{Z}_\ell} d\tilde{\rho}(z) \ f(z) = \int_{\Omega_c} dt \ \rho(t) \int_{K} dk \ f(k\sqrt{t}).$$

For $\ell = 1$ this simplifies to

$$\int_{\mathcal{Z}_1} d\tilde{\rho}(z) \ f(z) = \int_{0}^{\infty} dt \ \rho(t) \int_{S_1} du \ f(u\sqrt{t}).$$

Define a Hilbert space of holomorphic functions

$$\mathcal{H}_\rho := \{ f : \mathcal{Z}_\ell \rightarrow \mathbb{C} \text{ holomorphic} : \int_{\mathcal{Z}_\ell} d\tilde{\rho}(z) \ |f(z)|^2 < +\infty \}.$$ 

**Theorem:** The Hilbert space $\mathcal{H}_\rho$ has the reproducing kernel

$$K_\rho(z, w) := \sum_{m \in \mathbb{N}_+^\ell} \frac{d_m}{\int_{\Omega_c} d\rho(t) E^m(t, e) E^m(z, w)} = \sum_{m \in \mathbb{N}_+^\ell} \frac{(d_\ell/\ell)m}{\int_{\Omega_c} d\rho(t) N_m(t)} \frac{d_m}{d_c m} E^m(z, w).$$
Every euclidean Jordan algebra $X$ has a **Jordan algebra determinant** $N(x)$ which is a $r$-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla x N}{N(x)}.$$  

For real or complex matrices, this is the usual determinant. For even anti-symmetric matrices, it is the Pfaffian. In the rank 2 case, $N(x_0, x) = x_0^2 - (x|x)$ is the Lorentz metric.

Let $\partial_N = N(\frac{\partial}{\partial x})$ be the associated constant coefficient differential operator on $X$. Then $N(x)\partial_N$ is a kind of Euler operator, of order $r$.

Applying these concepts to the Jordan algebra $Z_c^{(2)}$ of rank $\ell$, we define a universal differential operator

$$D_{\ell} := N_{\frac{a}{2}}^{\ell-r} \partial_N^{b} N_{\frac{a}{2}}^{\ell-\ell+1+b+1} \left( \partial_N N_{\frac{a}{2}}^{\ell-1} \partial_N^{\ell-1} \right)^{r-\ell} N_{\frac{a}{2}}^{\ell-\ell+1-b-1}$$

of order $\ell((r-\ell)a+b)$ on the symmetric cone $D_c$. For $\ell = 1$, $N(t) = t$ and $D_1$ becomes a polynomial differential operator of order $(r-1)a+b$ on the half-line.
Our main result is that the reproducing kernel $K_\rho(z, w)$ can be expressed in closed form, by

$$K_\rho(\sqrt{t}, \sqrt{t}) = (D_\ell F_\rho)(t)$$

for all $t \in D_c$, where $F_\rho$ is a special function of hypergeometric type. Since hypergeometric functions have good asymptotic expansions (quite difficult for $\ell > 1$), this will lead to the desired TYZ-expansions. Generalizing the 1-dimensional case,
Consider first a Fock type situation. For $\alpha > 0$ consider the 'pluri-subharmonic' function $\phi = (z|z)^{\alpha}$ on $Z_{\ell}$. Let $\omega := \partial \overline{\partial} \phi$ denote the associated Kähler form. Then we have the polar decomposition

$$\int_{Z_{\ell}} \frac{\omega^n}{n!} (z) \ e^{-\nu \phi(z)} f(z) = \int_{D_c} dt \ N_c(t)^{a(r-\ell)+b} (t|c)^{n(\alpha-1)} \ e^{-\nu (t|c)^{\alpha}},$$

where $N_c(t)$ is the rank $\ell$ determinant function on $D_c$. Let $\alpha = 1$.

**Theorem:** The Hilbert space $\mathcal{H}_\nu = H^2(e^{-\nu \phi}|\omega^n|/n!)$ has the reproducing kernel

$$K_\nu(t, e) = D_\ell \ _1F_1\left(\frac{d_{\ell}/\ell}{n/\ell}\right)(\nu t)$$

for the **confluent hypergeometric function**

$$\ _1F_1\left(\frac{\sigma}{\tau}\right)(z, w) = \sum_m \frac{(\sigma)_m}{(\tau)_m} E^m(z, w).$$
In the bounded setting (Kepler ball) consider the pluri-subharmonic function

$$\phi(z) = \log \Delta(z, z)^{-p} = \log \det B(z, z)^{-1}$$

on the Kepler ball, given by the Bergman kernel. As before, the associated radial measure $e^{-\nu \phi} |\omega^n|/n!$ has a nice polar decomposition, this time involving the unit interval $D_c \cap (c - D_c)$.

**Theorem:** For the Kepler ball, $\mathcal{H}_\nu = H^2(e^{-\nu \phi} |\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t, e) = D_\ell \, _2F_1\left(\frac{d_\ell}{\ell}; \nu \frac{n_\ell}{\ell}\right)(t)$$

involving a *Gauss hypergeometric function* $\, _2F_1$ on $\Omega_c$. 
Special case, Lie ball $r = 2, \ell = 1$ (Englis et al, Gramchev-Loi)
non-compact Kähler manifold, homogeneous, non-symmetric

$$T^*(S^n) \setminus \{0\} = \{(x, \xi) : \|x\| = 1, (x|\xi) = 0, \xi \neq 0\}$$

$\mathbb{C}^{n+1}$ hermitian Jordan triple, rank 2

$$\{uv^* w\} = (u|v)w + (w|v)u + (u|\overline{w})\overline{v}$$ Jordan triple product

$$N(z) = (z|\overline{z})$$ Jordan algebra determinant

$$T^*(S^n) \setminus \{0\} = \{z \in \mathbb{C}^{n+1} : N(z) = 0\}$$ rank 1 elements

$$\frac{T_\nu(z)}{\nu^n} = 1 + \frac{(n-1)(n-2)}{2|\nu z|} + \sum_{j=2}^{n-2} \frac{2a_j}{|\nu z|^j} + R_\nu(|z|)$$ exponentially small error term
For $\ell = 1$ (minimal Kepler varieties) consider Fock space asymptotic expansion, with $\alpha$ arbitrary

**Theorem:** Radial measure $d\rho(t) = \alpha e^{-\nu t^\alpha} t^{\alpha(p-1)-1} \, dt$ on half-line. Then, as $\nu \to +\infty$

$$e^{-\nu(z|z)^\alpha} \frac{1}{\nu^{p-1}} K_\nu(z, z) = \sum_{j=0}^{p-2} b_j \frac{\nu^j (z|z)^{j\alpha}}{\nu j^{\alpha}} + O(e^{-\eta \nu(z|z)^\alpha}), \quad \eta > 0$$

Proof: The measure $d\rho$ has moments

$$\alpha \int_0^\infty dt \ e^{-\nu t^\alpha} t^{\alpha(p-1)+m-1} = \frac{\Gamma(p - 1 + \frac{m}{\alpha})}{\nu^{p-1+\frac{m}{\alpha}}}.$$ 

Therefore $K_\nu = D_1 F_\rho$ with

$$F_\rho(t) = \nu^{p-1} \sum_{m=0}^{\infty} \frac{(\nu^{1/\alpha} t)^m}{\Gamma(p - 1 + \frac{m}{\alpha})} = \nu^{p-1} E_{1/\alpha, p-1}(\nu^{1/\alpha} t),$$

for the **Mittag-Leffler function**

$$E_{A, B}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(Am + B)}.$$
Theorem:

For \( t \in D_c \) define the function
\[
F_\rho(t) = \sum_{m \in \mathbb{N}_+} \left( \frac{d_\ell/\ell}{\ell} \right)^m \int_{D_c} d\rho \, N_m(t, c).
\]

Every euclidean Jordan algebra \( X \) has a **positive cone**
\[
\Omega = \{ x^2 : x \in X \setminus \{0\} \},
\]
corresponding to the positive definite matrices \( \mathcal{H}^+_r(K) \). The **Jordan algebra determinant** \( N : X \to \mathbb{R} \) is a \( r \)-homogeneous polynomial defined via Cramer’s rule
\[
x^{-1} = \frac{\nabla_x N}{N(x)}.
\]

For the rank 2 case,
\[
N \begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} = \alpha \delta - (b|b)
\]
is the Lorentz metric on \( \mathbb{R}^{1,1+a} \). The complexification \( Z := X^\mathbb{C} \) becomes a (complex) Jordan triple (of tube type)
\[
\{uv^*w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).
\]

For the spin factor we obtain...
\[
\phi = (z | z)^\alpha, \quad \alpha > 0 \text{ pluri-subharmonic function on } Z_\ell \\
\omega := \partial \bar{\partial} \phi \text{ K"ahler form}
\]

\[
\int_{\hat{Z}_\ell} \frac{|\omega^n|}{n!}(z) e^{-\nu \phi(z)} f(z) = \int_{D_c} dt \ N_c(t)^{a(r-\ell)+b} (t|c)^n(\alpha-1) e^{-\nu(t|c)^\alpha}.
\]

**Theorem:** \( \mathcal{H}_\nu = H^2(e^{-\nu \phi}|\omega^n|/n!) \) has kernel \( \mathcal{K}_\nu(\sqrt{t}, \sqrt{t}) = D_\ell F_\nu(t) \)

\[
F_\nu(t) = \text{const} \sum_{m \in \mathbb{N}_+} \frac{(d_\ell/\ell)_m}{\Gamma_D(c)(m + n/\ell)} \frac{\Gamma(n + |m|)}{\Gamma(n + \frac{|m|}{\alpha})} E^m(\nu^{1/\alpha} t, c)
\]

confluent hypergeometric function

\[
1F_1(z, w) = \sum_m \frac{(\sigma)_m}{(\tau)_m} E^m(z, w)
\]

For \( \alpha = 1 \)

\[
F_\nu(t) = 1F_1(d_\ell/\ell)(\nu t)
\]

**Theorem:** \( \mathcal{H}_\nu = H^2(e^{-\nu \phi}|\omega^n|/n!) \) has kernel \( \mathcal{K}_\nu(\sqrt{t}, \sqrt{t}) = D_\ell F_\nu(t) \)

\[
F_\nu(t) = \text{const} \sum_{m \in \mathbb{N}_+} \frac{(d_\ell/\ell)_m}{\Gamma_D(c)(m + n/\ell)} \frac{\Gamma(n + |m|)}{\Gamma(n + \frac{|m|}{\alpha})} E^m(\nu^{1/\alpha} t, c)
\]
The invariant probability measure $dg$ on $G^+$ satisfies

$$\int_{G^+} dg f(g(0)) = \frac{\Gamma_O(p)}{\Gamma_O(p - \frac{d}{r})} \int_Z \frac{d\overline{w}dw}{(2\pi i)^d} \Delta^{-p}_{w,-w} f(w)$$

for all $f \in L^\infty(Z)$. Here $dw$ denotes the Lebesgue measure for the inner product $(.)$, and the normalization constant is computed in terms of the Gindikin $\Gamma_O$-function.
Spectral Analysis Since $Z^+ = G^+/K$ is a compact symmetric space, the space $L^2(Z^+)$, for the normalized Haar measure (1.), has a multiplicity-free “Peter-Weyl” decomposition into irreducible $G^+$-submodules. This “Peter-Weyl” decomposition can be described in a uniform manner:

$$L^2(Z^+) = \sum_m L^2_m(Z^+) \tag{0.1}$$

where $m$ runs over all partitions of length $r$, and $L^2_m(Z^+)$ is the (finite-dimensional) $G^+$-submodule generated by the spherical function $\Phi^+_m$. 
example

In the rank 1 case $Z^+ = \mathbb{CP}^1 \approx S^2$, $L_m^2(Z^+)$ is spanned by the functions

$$z \mapsto \left[ \frac{\alpha z + \beta \overline{z} + \gamma (z \overline{z} - 1)}{1 + |z|^2} \right]$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ satisfy $\alpha, \beta + 4\gamma^2 = 0$. The corresponding spherical function is

$$\Phi_m^+(z) = C_m \left( \frac{1 - |z|^2}{1 + |z|^2} \right)^m,$$

where $C_m$ is the $m$-th Gegenbauer polynomial.
Invariant volume forms and measures

\( \mathring{Z}_\ell \) is homogeneous under \( K^C \) or a suitable transitive subgroup. When does an invariant holomorphic volume form or invariant measure exist? Only for domains of tube type \( b = 0 \).

- **Theorem:** Let \( Z \) be of tube type and \( 0 < \ell < r \). Then there exists an invariant holomorphic \( n \)-form \( \Omega \) on \( \mathring{Z}_\ell \) if and only if \( p - a\ell = 2 + a(r - \ell - 1) \) is even.

- Among all tube type domains, \( p - a\ell \) is odd only for the symmetric matrices \( Z = C^{r \times r}_{\text{sym}} \) with \( r - \ell \) even.

- **Theorem:** An invariant measure \( \mu \) on \( \mathring{Z}_\ell \) exists in all cases (for \( b = 0 \)), and has polar decomposition

\[
\int_{\mathring{Z}_\ell} d\mu(z) \ f(z) = \text{const} \int_{D_c} \frac{dt}{N_c(t)d_{\ell/\ell}} \ N_c(t)^{ar/2} \int_K dk \ f(k\sqrt{t}).
\]