Geometry and Analysis on Kepler Manifolds

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5. Juni 2019

[A]

A **hermitian Jordan triple** is a complex vector space Z, endowed with a ternary composition $Z \times Z \times Z \to Z$, denoted by

$$(x, y, z) \mapsto \{x; y; z\},$$

which is bilinear symmetric in (x, z) and anti-linear in the inner variable, and satisfies the Jordan triple identity

$$[L(x, y), L(u, v)] = L(\{x; y; u\}, v) - L(u, \{v; x; y\})$$

where

$$L(x, y)z := \{x; y; z\}.$$

Moreover, the hermitian form

$$(x, y) \mapsto \text{trace } L(x, y)$$

is positive definite. Let K denote the compact group of all linear transformations of Z preserving the triple product

$$k\{u; v; w\} = \{ku; kv; kw\}.$$

Define the **quadratic representation**

$$Q_z w := \frac{1}{2}\{z; w; z\}$$

and the **Bergman endomorphisms**

$$B(u,v)z = z - \{u; v; z\} + \frac{1}{4}\{u; \{v; z; v\}; u\}$$

for $u, v, z \in Z$. Thus

$$B(u,v) = \mathrm{id} - L(u,v) + Q_u Q_v$$

This is a complex-linear endomorphism of Z, although Q_u and Q_v are conjugate-linear. Later we will use the **Jordan triple determinant**

$$\Delta(u,v) = \det\ B(u,v)^{1/p}$$

Consider the **matrix triple** $Z = \mathbf{C}^{r\times s}$, with $r \le s$.

$$\{x;y;z\} = xy^*z + zy^*x$$

$$Q_z w = zw^*z$$

$$B(u,v)z = z - uv^*z - zv^*u + u(vz^*v)^*u = (1-uv^*)z(1-v^*u)$$

$$B(u,v) = L_{1-uv^*}\ R_{1-v^*u}.$$

$$\Delta(z,w) = \det\ (I_r - zw^*) = \det\ (I_s - w^*z)$$

$$K = U(r)\times U(s):\ z\mapsto uzv,\ u\in U(r),\ v\in U(s)$$

$$rank = r \le s,\ a = 2 \text{ complex case},\ b = s-r$$

$$r = 1,\ Z = \mathbf{C}^{1\times d},\ \{x;y;z\} = (x|y)z + (z|y)x$$

$$K = U(d) \text{ unitary group}$$

$$d = 1,\ Z = \mathbf{C},\ \{x;y;z\} = 2x\overline{y}z$$

$$K = U(1) \text{ 1-torus}$$

A real vector space X is called a **Jordan algebra** iff X has a non-associative product $x, y \mapsto x \circ y = y \circ x$, satisfying the Jordan algebra identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y).$$

The **anti-commutator product**

$$x \circ y = (xy + yx)/2$$

of self-adjoint operators on a Hilbert space satisfies the Jordan identity. Every euclidean Jordan algebra X has a **symmetric cone**

$$\Omega = \{x^2 : \ x \in X \text{ invertible}\}.$$

By a fundamental result of Jordan/v. Neumann/Wigner (1934), every (euclidean) Jordan algebra have a **classification** as **self-adjoint matrices**

$$X \approx \mathcal{H}_r(\mathbf{K}) = \{(x_{ij}) \in \mathbf{K}^{r\times r},\ x_{ij}^* = x_{ji}\}$$

endowed with the anti-commutator product. Here $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ (quaternions), or $\mathbf{K} = \mathbf{O}$ (octonions) if $r \le 3$. For $r = 2$ we obtain formal 2×2-matrices **real spin factor**

$$\mathcal{H}_2(\mathbf{K}) = \{\begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} :\ \alpha, \delta \in \mathbf{R},\ b \in \mathbf{K} := \mathbf{R}^{d-2}\}.$$

Thus X is characterized by the **rank** r and the **multiplicity**

$$a = \dim_{\mathbf{R}}\ \mathbf{K}$$

The symmetric cone Ω corresponds to the positive definite matrices $\mathcal{H}_r^+(\mathbf{K})$.

For a (euclidean) Jordan algebra X, the **complexification**

$$Z := X^{\mathbf{C}} = X \oplus iX$$

becomes a hermitian Jordan triple via

$$\{u;v;w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

The Jordan triples arising this way are called of **tube type**. For example, the matrix triple $\mathbf{C}^{r \times s}$ is of tube type if and only if $r = s$. For the spin factor of rank $r = 2$ we obtain $Z = \mathbf{C}^d$ with triple product

$$\{u;v;w\} = (u|v)w + (w|v)u - \overline{v}(u|\overline{w}).$$

$$K = \mathbf{T} \cdot SO(n+1)$$

Classification of hermitian Jordan triples:

- **matrix triple** $Z = \mathbf{C}^{r\times s}$ $\mathrm{rank} = r \leq s,\ a = 2$ complex case, $b = s - r$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- **spin factor** $Z = \mathbf{C}^{n+1},\ r = 2,\ a = n - 1,\ b = 0$
- **exceptional Jordan triples** of dimension 16 ($r = 2$) and 27 ($r = 3$), $a = 8$ (octonion case), $K = \mathbf{T} \cdot E_6$.

$$d := \dim Z = r\left(1 + \frac{a}{2}(r-1) + b\right)$$

$$r = \mathrm{rank}(Z),$$

a, b characteristic multiplicities

$b = 0$ tube type

[B]

For every hermitian Jordan triple Z the **spectral unit ball**

$$D = \{z \in Z : B(z,z) > 0\} = \{z \in Z : \|z\|_\infty < 1\}$$

is a symmetric domain in its Harish-Chandra realization. This means that the group $G = \mathrm{Aut}(D)$ of all biholomorphic automorphhisms of D acts transitively on D. The stabilizer subgroup K at the origin $0 \in D$ consists of all linear transformations preserving the Jordan triple product. A basic theorem of M. Koecher asserts that, conversely, every symmetric domain can be realized as the unit ball of a unique hermitian Jordan triple Z. For matrices we obtain the matrix ball

$$D = \{z \in \mathbf{C}^{r\times s} : \ I_r - zz^* > 0\}.$$

The domain D (or the Jordan triple Z) is said to be of **tube type** if D is biholomorphically equivalent to a tube domain over a symmetric cone Ω.

The G-invariant **Bergman metric** on the tangent spaces $T_z D = Z$ is given by

$$(u|v)_z = (B(z,z)^{-1} u|v)_0,$$

using the K-invariant inner product $(u|v)_0$ on $T_0 D = Z$ given by

$$(u|v)_0 := \mathrm{tr}\ L(u,v).$$

For matrices, we have

$$(u|v)_0 = \mathrm{tr}\ L(u,v) = \mathrm{trace}\ (L_{uv^*} + R_{v^*u}) = (r+s)\ \mathrm{tr}\ uv^*,$$

and the Bergman metric at $z \in D$ is

$$(u|v)_z = (r+s)\ \mathrm{tr}\ (1 - zz^*)^{-1}\ u(1 - z^*z)^{-1}\ v^*.$$

Boundary of symmetric domains

An element $c \in Z$ is called a **tripotent** if

$$\{c;c;c\} = 2c$$

or, equivalently, $Q_c c = c$. For $Z = \mathbf{C}^{r\times s}$ tripotents satisfy $cc^*c = c$ and are called partial isometries. Every tripotent c induces a **Peirce decomposition**

$$Z = Z_2^c \oplus Z_1^c \oplus Z_0^c,$$

where

$$Z_j^c = \{z \in Z : \ \{c;c;z\} = 2jz\}$$

is an eigenspace of $L(c,c)$. Moreover, the Peirce 2-space Z_2^c becomes a Jordan *-algebra with unit element c, multiplication

$$(z,w) \mapsto \frac{1}{2}\{z;c;w\}$$

and involution

$$z \mapsto z^* = \frac{1}{2}\{c;z;c\} = Q_c z$$

The set of all tripotents is a (non-connected) real-analytic compact manifold, whose connected components

$$S_\ell := \{c \in Z \text{ tripotent}: \text{ rank } c = \ell\},$$

for $0 \leq \ell \leq r$, are K-homogeneous. Here the **rank** of a tripotent c is the rank of its Peirce 2-space Z_2^c. We have $S_0 = \{0\}$ and S_1 consists of all **minimal** (rank 1) tripotents. The **maximal** tripotents

$$S := S_r = \partial_{ex} D$$

form the **Shilov (extreme) boundary** of D.

One can show that the spectral norm satisfies

$$\|u+w\| = \max(\|u\|, \|w\|)$$

whenever $u \in Z_2^c,\ w \in Z_0^c$. Therefore, if $c \neq 0$, the set

$$c + D_0^c := \{c + w : \ w \in D \cap Z_0^c\}$$

belongs to the boundary ∂D, since $\|c\| = 1$. These are the so-called **boundary components** of D. They are pairwise disjoint and cover the whole boundary. The boundary components are precisely the (open) faces of ∂D in the sense of convex geometry. Thus we have a disjoint union

$$\partial D = \dot{\bigcup}(c + D_0^c).$$

over all non-zero tripotents c. The tripotent $c = 0$ gives the interior D.

For any $1 \leq k \leq r$ the disjoint union

$$\partial_k D = \dot{\bigcup}_{c \in S_k} (c + D_0^c)$$

is a G-orbit contained in the boundary. Thus

$$\partial D = \dot{\bigcup}_{1 \leq k \leq r} \partial_k D$$

is a disjoint union of G-orbits, called the **boundary orbits**. If $k = r$ then

$$\partial_r D = S_r = S$$

is the **Shilov boundary** (realized as a G-orbit $S = G/P$) and the corresponding boundary components are the extreme points, since $Z_0^c = (0)$. For higher rank, the boundary is not smooth.

In the rank 1 case $Z = \mathbf{C}^{1\times d}$

$$S_1 = \{u \in Z : \ (u|u) = 1\} = \mathbf{S}^{2d-1} = S$$

is the full boundary. For the spin factor $Z = \mathbf{C}^d$ of $r = 2$, we have

$$\{uv^*w\} = (u|v)w + (w|u)v - (u|\overline{w})\overline{v}.$$

$$S = S_2 = \mathbf{T} \cdot \mathbf{S}^{d-1}$$

is called the **Lie sphere**, and the minimal tripotents

$$S_1 \approx S^*(\mathbf{S}^n) = \{(x,\xi) : \|x\| = 1, \ \|\xi\| = 1, \ (x|\xi) = 0\}$$

form the **cosphere bundle** of $\mathbf{S}^n$, which is a contact manifold.

Kepler varieties

A Jordan algebra X has a **determinant function** $N(x)$ satisfying Cramer's rule

$$x^{-1} = \frac{\mathrm{grad}_x N}{N(x)}.$$

The Jordan algebra determinant $N : X \to \mathbf{R}$ is a **homogeneous** polynomial of degree r. For square matrices over $\mathbf{K} = \mathbf{R}, \mathbf{C}$ we have

$$N(x) = \det(x).$$

For the rank 2 case,

$$N\begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} = \alpha\delta - (b|b)$$

is the Lorentz metric on $\mathbf{R}^{1,d-1}$.

For $\ell \leq r$ define the **Kepler variety**

$$Z_\ell = \{z \in Z: \text{ rank}(z) \leq \ell\}.$$

Here $\text{rank}(z) \leq \ell$ if $N(z) = 0$ for all Jordan determinants of degree $> \ell$. Its regular part (**Kepler manifold**)

$$\mathring{Z}_\ell = \{z \in Z_\ell: \text{ rank}(z) = \ell\}$$

is a $K^{\mathbf{C}}$-homogeneous manifold, not compact or symmetric. The Kepler manifolds $\mathring{Z}_\ell$ are precisely the **Matsuki dual** $K^{\mathbf{C}}$-orbits of the boundary G-orbits $\partial_\ell D$, with common intersection

$$\mathring{Z}_\ell \cap \partial_\ell D = S_\ell$$

being the K-orbit of tripotents.

For matrix spaces $Z = \mathbf{C}^{r\times s},\ r \le s$ and $1 \le \ell \le r$, we obtain the **determinantal varieties**

$$Z_\ell := \{z \in Z :\ \mathrm{rank}(z) \le \ell\}$$

defined by vanishing of all $(\ell+1)\times(\ell+1)$-minors. The regular part is

$$\mathring{Z}_\ell = \{z \in Z :\ \mathrm{rank}(z) = \ell\} = Z_\ell \setminus Z_{\ell-1}.$$

In particular

$$Z_1 := \{z \in Z :\ \mathrm{rank}(z) \le 1\} \quad \text{vanishing of all } 2\times 2 - \text{minors}$$

$$\mathring{Z}_1 := \{z \in Z :\ rank(z) = 1\} = Z_1 \setminus \{0\}$$

$$S_1 = \{\xi \otimes \eta^* :\ \xi \in \mathbf{C}^r, \eta \in \mathbf{C}^s,\ \|\xi\| = 1 = \|\eta\|\}$$

For the spin factor $Z = \mathbf{C}^{n+1}$ of $r = 2$, the Jordan determinant

$$N(z) = \frac{1}{2}(z|\overline{z})$$

is the (complexified) Lorentz metric, and we recover the classical Kepler variety

$$Z_1 = \{z \in \mathbf{C}^d : N(z) = 0\} = \{z = (z_0, \ldots, z_n) \in Z : z_0^2 + \ldots + z_n^2 = 0\}$$

(complex light cone). Symplectic interpretation as cotangent bundle

$$\mathring{Z}_1 \approx T^*(\mathbf{S}^n) \setminus \{0\} = \{(x,\xi) : \|x\| = 1,\ \xi \neq 0,\ (x|\xi) = 0\},\ z = \frac{x + i\xi}{2}$$

$$n = \dim_{\mathbf{C}} Z_1 = p - 1 = d - 1$$

$$S_1 = \{\frac{x + i\xi}{2} : |x| = |\xi| = 1, x \cdot \xi = 0\} = \mathbf{S}^*(\mathbf{S}^{d-1})$$

cosphere bundle.

$$\dim S_1 = 2(d-1) - 1 = 2n - 1$$

In joint work with M. Englis, we have studied reproducing kernels and their asymptotic expansion on Kepler manifolds.

Theorem: The Kepler variety Z_ℓ is a normal variety having only rational singularities.

This result is classical for spin factors $(r = 2)$ and proved by Kaup-Zaitsev for non-exceptional types. Our proof is uniform, based on Kempf's collapsing vector bundle theorem.

[C]

Two tripotents $u, v \in Z$ of the same rank ℓ are said to be **Peirce equivalent** if

$$Z_\lambda^u = Z_\lambda^v$$

for all $\lambda = 0, 1, 2$. It is enough to consider the Peirce 2-space, since this determines the other two Peirce subspaces. The ℓ-th **Peirce manifold** M_ℓ associated with the Jordan triple Z is defined as the quotient space

$$M_\ell := S_\ell / \sim$$

of S_ℓ under the Peirce equivalence relation. Thus M_ℓ is the set of all Peirce 2-spaces $U \subset Z$ having rank ℓ, in analogy with the classical Grassmann manifolds. It is known that M_ℓ is a compact hermitian symmetric space for the semi-simple compact Lie group $\dot{K}$ generated by the commutators in K. Being a complex manifold, M_ℓ is both a K-orbit and a $K^{\mathbf{C}}$-orbit. The Peirce manifolds are the **Matsuki duals** of the open G-orbits (in general non-convex) in the conformal compactification $\hat{Z}$.

Clearly, M_0 consists only of $\{0\}$ and in the tube case $Z = X^{\mathbf{C}}$ $M_r = \{Z\}$, since Z is the only Peirce 2-space of maximal rank. For non-tube domains, M_r has higher dimension. For example, if $r = 1$, i.e., $Z = \mathbf{C}^{1\times d}$ viewed as row vectors, M_1 coincides with complex projective space $\mathbf{P}^{d-1}$. More generally, for $c \in S_\ell$ we may identify

$$M_\ell = \hat{Z}_1^c$$

with the compact dual space (conformal compactification) for the Peirce 1-space Z_1^c, viewed as a hermitian Jordan subtriple of Z.

The first class of non-tube type Jordan triples are the **rectangular matrix spaces**

$$Z = \mathbf{C}^{r\times(r+b)}$$

of rank r, where $b > 0$. Choosing the maximal tripotent $e = (1_r, 0)$ of rank r, we obtain the Peirce 1-space

$$Z_1^e = \{(0, w) |\ w \in \mathbf{C}^{r\times b}\} \approx \mathbf{C}^{r\times b}$$

Its conformal hull gives the Grassmann manifold

$$M_r = \hat{Z}_1^e = \mathrm{Grass}_r(\mathbf{C}^{r+b})$$

of all r-dimensional subspaces in $\mathbf{C}^{r+b}$.

In the special case $r = 1$, corresponding to the unit ball in $Z = \mathbf{C}^{1+b},\ e = (1,0)$ is a unit vector and we have $Z_1^e = \mathbf{C}^b$.. Thus

$$M_1 = \mathrm{Grass}_1(\mathbf{C}^{1+b}) = \mathbf{P}^b(\mathbf{C})$$

is the projective space. This is **not** the compact dual of the unit ball, which is

$$\hat{Z} = \mathrm{Grass}_1(\mathbf{C}^{2+b}) = \mathbf{P}^{1+b}(\mathbf{C}).$$

For $U \in \mathbf{G}_j(Z)$, the fibre

$$\pi_j^{-1}(U) = \mathbf{S}(U) = \mathbf{S}_j(Z) \cap U$$

coincides with the Shilov boundary of the unit ball $D \cap U$. One can show that different Peirce 1-spaces U and V of the same rank have *disjoint* Shilov boundaries:

$$U \neq V \implies \mathbf{S}(U) \cap \mathbf{S}(V) = \varnothing$$

We call the disjoint union

$$\mathbf{S}_j(Z) = \bigcup_{U \in \mathbf{G}_j(Z)} \mathbf{S}(U),$$

the *tautological bundle* over $\mathbf{G}_j(Z)$. Note that in Section 2, we realized $\mathbf{S}_j(Z)$ as the base manifold of a bundle instead.

The Jordan triple automorphism group of Z is given by

$$K = \mathrm{Aut}(Z) = U(r) \times U(r+b)$$

acting on Z by left and right multiplication $(a,d)(z) := azd^{-1}$ for $a \in U(r),\ d \in U(r+b)$ and $z \in \mathbf{C}^{r\times(r+b)}$. The complexification $K^{\mathbf{C}} = GL_r(\mathbf{C}) \times GL_{r+b}(\mathbf{C})$ acts in the same way. It follows that the action of K on $\mathbf{G}_r(Z)$ can be realized as the **collineation action**

$$K \xrightarrow{pr_2} U(r+b) \longrightarrow \mathrm{Aut}(\mathrm{Grass}_r(\mathbf{C}^{r+b}))$$

of the second factor.

Define the **tautological vector bundle**

$$\mathcal{T}_\ell := \bigcup_{U \in M_\ell} U \to M_\ell.$$

Fix a base point $c \in S_\ell$, put $V := Z_2^c$ and let $L := \{k \in K^{\mathbf{C}} : \ kV = V\}$. Then $\mathcal{T}_\ell = K^{\mathbf{C}} \times_L V$ becomes a **homogeneous vector bundle**, and

$$\mathcal{T}_\ell \to Z_\ell : \ U \ni z \mapsto z \in Z_\ell$$

is a collapsing map. By **Kempf's Theorem**, the range of a collapsing map of a homogeneous vector bundle is a normal variety. Let $\mathring{U}$ denote the invertible elements of the Jordan algebra $U = Z_2^u$. Then

$$\bigcup_{U \in M_\ell} \mathring{U} \to Z_\ell$$

is a birational isomorphism. This proves the normality theorem.

For any $\ell \leq r$, the **Kepler ball** is the intersection

$$\{z \in \mathring{Z}_\ell : \ \|z\|_\infty < 1\} = \mathring{Z}_\ell \cap D$$

of $\mathring{Z}_\ell$ with the bounded symmetric domain $D := \{z \in Z : \ \|z\|_\infty < 1\}$. The compact K-homogeneous manifold S_ℓ of all rank ℓ tripotents can be regarded as the Shilov boundary of the Kepler ball D_ℓ. It carries the normalized K-invariant measure

$$d\mu_\ell(u) = du.$$

Let $B(u,v)$ denote the Bergman operators on a hermitian Jordan triple Z.

Proposition For any tripotent $c \in S_\ell$, the holomorphic map

$$Z_1^c \to M_\ell, \quad v \mapsto [B(v,-c)c]$$

is a local chart of M_ℓ. Here $[z]$ means the Peirce 2-space of an element $z \in \mathring{Z}_\ell$ of rank ℓ.

Proposition For any tripotent $c \in S_\ell$, the holomorphic map

$$Z_2^c \times Z_1^c \to \mathcal{T}_\ell, \quad (u,v) \mapsto B(v,-c)u$$

becomes a vector-bundle chart of $\mathcal{T}_\ell$.

The above construction generalizes the well-known charts

$$\sigma_j(v_0, \ldots \hat{v}_j, \ldots, v_d) := [v_0 : \ldots : 1 :, \ldots, : v_d] \in \mathbf{P}^d$$

for projective space, where $Z = \mathbf{C}^d$ and $0 \leq j \leq d$. In fact, putting $j = 0$ for simplicity, we choose $c = (1, 0, \ldots, 0) \in S_1$, with Peirce 1-space $Z_1^c = (0, \mathbf{C}^{d-1})$. Then for all $v \in \mathbf{C}^{d-1}$ we have

$$B(v, -c)c = (1 + (v|c))c(I_d + c^* v) = (1, 0) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0, v) \right) = (1, v).$$

Passing to homogeneous coordinates we obtain $[1 : v] = \sigma_0(v)$.

The classical **blow-up process** used in algebraic geometry replaces a point in a complex manifold of dimension n by a projective space $\mathbf{P}^{n-1}$. It is based on the incidence space

$$\mathcal{T} := \{(z, U) : \ z \in \mathbf{C}^n, \ U \in \mathbf{P}^{n-1}, \ z \in U\}$$

and the projection map

$$\pi : \mathcal{T} \to \mathbf{C}^n, \quad (z, U) \mapsto z,$$

whose fibre over $z \neq 0$ is the singleton $(z, [z])$, whereas the fibre over $0 \in \mathbf{C}^n$ is $\mathbf{P}^{n-1}$, regarded as a compact hermitian symmetric space of rank 1. For the Kepler varieties, we construct a blow-up process using compact hermitian symmetric spaces of higher rank.

Theorem The Kepler variety Z_ℓ of all elements of rank $\leq \ell$ has the regular (smooth part)

$$\mathring{Z}_\ell = \{z \in Z : \ \mathrm{rank}(x) = \ell\}.$$

Thus $Z_{\ell-1}$ is exactly the singular part of Z_ℓ. Consider the **incidence space**

$$\mathcal{T}_\ell = \{(z, U) \in Z_\ell \times M_\ell : \ z \in U\}$$

endowed with the local charts constructed via the Bergman operators. Then the projection

$$\pi : \mathcal{T}_\ell \to Z_\ell, \quad (z, U) \mapsto z,$$

is a proper holomorphic modification in the sense that for $z \in \mathring{Z}_\ell$ the fibre of π is the singleton (z, Z_2^z), whereas for $z \in Z_{\ell-1}$ the fibres have higher dimension, in particular, the fibre over 0 is the full Peirce manifold M_ℓ, which is a compact hermitian space of rank ℓ.

[E]
[D]

For a hermitian Jordan triple Z, consider the algebra $\mathcal{P}(Z)$ of all polynomials $p : Z \to \mathbf{C}$ and the natural action

$$(k^{-1}\ p)(z) := p(kz)$$

of $K = \mathrm{Aut}(Z)$. Then there is a multiplicity-free **Hua-Schmid-Kostant decomposition**

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^r} \mathcal{P}_{\boldsymbol{m}}(Z)$$

into pairwise inequivalent irreducible K-modules $\mathcal{P}_{\boldsymbol{m}}(Z)$, where

$$\boldsymbol{m} = m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$$

ranges over all **integer partitions** (Young diagrams). For $r = 1$ (Hilbert ball) $\mathcal{P}_m(Z)$ consists of all m-homogeneous polynomials.

The highest weight vector $N_{\boldsymbol{m}} \in \mathcal{P}_{\boldsymbol{m}}(Z)$ can be described in terms of Jordan theoretic minors. Let $e_1, \ldots, e_r$ be a frame of minimal tripotents of Z, with joint Peirce decomposition

$$Z = \sum_{0 \leq i \leq j \leq r} Z_{ij}.$$

Then $\mathcal{P}_{\boldsymbol{m}}(Z)$ has the highest weight vector

$$N_{\boldsymbol{m}}(z) = N_1(z)^{m_1 - m_2} \; N_2(z)^{m_2 - m_3} \; N_r(z)^{m_r},$$

where $N_1, \ldots, N_r$ are the Jordan theoretic minors. For the simplest partition $\boldsymbol{m} = (1, 0, \ldots, 0)$, $\mathcal{P}_{\boldsymbol{m}}(Z)$ is the dual space of linear forms on Z. In this case $N_{\boldsymbol{m}}(z) = (e_1|z)$.

In the trivial case, where $\ell = r$ is maximal and Z is a Jordan algebra, negative powers of the (Jordan) determinant have no holomorphic extension beyond $\mathring{Z}$. Apart from this we have

Extension Theorem: Every holomorphic function on $\mathring{Z}_\ell$ has a unique extension to the closure Z_ℓ

Proof: Singular set

$$Z_\ell \setminus \mathring{Z}_\ell = \bigcup_{j<\ell} Z_j$$

has codimension ≥ 2. By the Normality Theorem (and using Lojasiewicz), the **second Riemann extension theorem** holds for holomorphic functions on $\mathring{Z}_\ell$. As a consequence, all K-invariant Hilbert spaces of holomorphic functions on Z_ℓ are completions of

$$\mathcal{P}^\ell(Z) = \sum_{m_1,\ldots,m_\ell,0,\ldots,0} \mathcal{P}_{\boldsymbol{m}}(Z)$$

involving only partitions of length $\leq \ell$.

The classical Fock (or Segal-Bargmann) space $H^2(Z)$ of entire functions on $Z = \mathbf{C}^d$ has the inner product

$$(\phi|\psi)_Z := \frac{1}{\pi^d} \int_Z dz\ e^{-(z|z)}\ \overline{\phi(z)}\ \psi(z).$$

The polynomials $\mathcal{P}(Z)$ are dense in $H^2(Z)$ and we have

$$(p|q)_Z = (\partial_p\ q)(0)$$

where ∂_p is the constant-coefficient differential operator associated with p via the normalized inner product. The multi-variable **Pochhammer symbol** is

$$(\nu)_{\boldsymbol{m}} = \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j} = \frac{\Gamma_\Omega(\nu + \boldsymbol{m})}{\Gamma_\Omega(\nu)}$$

where Γ_Ω denotes the Gindikin Gamma-function.

Let $\Sigma \subset Z$ be a K-invariant set of uniqueness for holomorphic functions. A K-invariant measure μ with support $\overline{\Sigma}$ is called **hypergeometric** of type $\binom{x_0,\ldots,x_h}{y_1,\ldots,y_h}$, if for all partitions $\boldsymbol{m}$ and all $\phi, \psi \in \mathcal{P}_{\boldsymbol{m}}(Z)$ we have

$$(\phi|\psi)_\mu := \int \mu(dz)\ \overline{\phi(z)}\ \psi(z) = \frac{(y_1)_{\boldsymbol{m}} \cdots (y_h)_{\boldsymbol{m}}}{(x_0)_{\boldsymbol{m}} \cdots (x_h)_{\boldsymbol{m}}}\ (\phi|\psi)_Z.$$

The measure is uniquely determined by this property, since the spaces $\mathcal{P}_{\boldsymbol{m}}(Z)$ are mutually orthogonal.
If $\Sigma \subset Z_\ell$ and the identity holds only for partitions $\boldsymbol{m}$ of length $\leq \ell$, we say that μ is ℓ-hypergeometric.
Finding a hypergeometric measure μ of a given type amounts to solving a **moment problem** involving all partitions $\boldsymbol{m}$.

For each partition $\boldsymbol{m}$ choose an orthonormal basis $p_\alpha \in \mathcal{P}_{\boldsymbol{m}}(Z)$ and define the **Fischer-Fock reproducing kernel**

$$E^{\boldsymbol{m}}(z,w) = \sum_\alpha p_\alpha(z)\overline{p_\alpha(w)}.$$

In the rank 1 case $Z = \mathbf{C}^{1\times d}$ we have

$$E^m(z,w) = \frac{(z|w)^m}{m!}.$$

The condition above is equivalent to saying that the Hilbert space $H^2_\mu(\Sigma)$ of all square-integrable holomorphic functions determined by μ has the **reproducing kernel**

$${}_q\mathcal{F}_p\binom{x_0,\dots,x_h}{y_1,\dots,y_h}(z,w) := \sum_{\boldsymbol{m}} \frac{(x_0)_{\boldsymbol{m}}\cdots(x_h)_{\boldsymbol{m}}}{(y_1)_{\boldsymbol{m}}\cdots(y_h)_{\boldsymbol{m}}}\ E^{\boldsymbol{m}}(z,w),$$

given by a **hypergeometric series**. For $z,w \in Z_\ell$, only partitions of length $\leq \ell$ occur.

- The Gauss measure

$$\mu(dz) = dz\ e^{-(z|z)}$$

on $\Sigma = Z$ is hypergeometric of type $\binom{}{}$, since the **Fock space** (or Segal-Bargmann space) $H^2_\mu(Z)$ of entire functions on Z has the ${}_0\mathcal{F}_0$-type reproducing kernel

$$e^{(z|w)} = \sum_{\boldsymbol{m}} E^{\boldsymbol{m}}(z,w).$$

- Let Δ be the Jordan triple determinant and $\nu > p-1$. The measure

$$\mu(dz) = dz\ \Delta(z,z)^{\nu-p}$$

on $\Sigma = D$ (bounded symmetric domain) is hypergeometric of type $\binom{\nu}{}$, since the **weighted Bergman spaces** $H^2_\mu(D)$ has the reproducing kernel

$$\Delta(z,w)^{-\nu} = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}}\ E^{\boldsymbol{m}}(z,w)$$

of type ${}_1\mathcal{F}_0$. This follows from the well-known **Faraut-Korányi formula**.

- Let $\Sigma = \partial_k D$ be the k-th boundary orbit $(1 \leq k \leq r)$. In joint work with J. Arazy, a measure μ on Σ was defined in terms of polar decomposition, which is hypergeometric of type $\binom{\nu_k}{}$, where

$$\nu_k = \frac{d}{r} + \frac{a}{2}(r-k)$$

are the so-called **embedded Wallach parameters**.

- The case $\nu_r = \frac{d}{r}$ corresponds to the **Hardy space**, where $\Sigma = S = \partial_r D$ is the Shilov (extreme) boundary and μ is the K-invariant probability measure.
- The case $\nu_1 = p - 1$, the left endpoint of the discrete series of weighted Bergman spaces, corresponds to the open boundary orbit $\partial_1 D$.

For the Kepler manifold $\ell \leq r$ we obtain (joint with M. Englis)

- Let $\Sigma = \mathring{Z}_\ell$ be the Kepler manifold, endowed with the K-invariant measure λ_ℓ induced by the Riemannian structure. Then the measure

$$\mu(dz) = e^{-(z|z)}\ \lambda_\ell(dz)$$

  on Σ is ℓ-hypergeometric of type $\binom{d_\ell/\ell}{n/\ell}$, where $n = \dim Z_\ell$ and $d_\ell = \dim Z_2^c$ for $c \in S_\ell$. Thus the Fock-type space $H^2_\mu(Z_\ell)$ has a reproducing kernel of type ${}_1\mathcal{F}_1$.

- Let $\Sigma = D \cap Z_\ell$ be the **Kepler ball**. Then the (restricted) measure

$$\mu(dz) = \Delta(z,z)^{\nu-p}\ d\lambda_\ell(z)$$

  is ℓ-hypergeometric of ${}_3\mathcal{F}_2$-type $\binom{r\frac{a}{2},\ \nu,\ \frac{d}{r}}{\ell\frac{a}{2},\ \nu_\ell}$

Finally, one may combine Kepler manifolds and boundary orbits (joint with G. Misra)

- The boundary orbit $\Sigma = \partial_k D \cap Z_\ell$ of the Kepler ball $(k \le \ell)$ supports a K-invariant measure μ which is ℓ-hypergeometric of type $\binom{r\frac{a}{2},\ \nu_k,\ \frac{d}{r}}{\ell\frac{a}{2},\ \nu_\ell}$.
- For integration over the **partial Shilov boundary** $\Sigma = S_\ell$ of $D \cap Z_\ell$ $(k = \ell)$ we obtain $\binom{r\frac{a}{2},\ \frac{d}{r}}{\ell\frac{a}{2}}$

Pattern

Fock space on $Z = \mathbf{C}^d,\ \mathcal{K}_\nu = e^{(z|w)} = {}_0\mathcal{F}_0(z,w)$

Bergman space on D, $\mathcal{K}_\nu = \Delta(z,w)^{-\nu} = {}_1\mathcal{F}_0(z,w)$ Faraut-Koranyi formula

Kepler manifold $\mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_1\mathcal{F}_1$

Kepler ball $D \cap \mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_2\mathcal{F}_1$

For a given measure μ on Σ, consider the Hilbert space $H^2_\mu(\Sigma)$ of all square-integrable holomorphic functions, restricted to Σ, and denote by $P_\mu : L^2_\mu(\Sigma) \to H^2_\mu(\Sigma)$ the orthogonal projection. For continuous symbol function $f \in \mathcal{C}(\Sigma)$ define the **Toeplitz operator**

$$T_f(\phi) := P_\mu(f\ \phi)$$

for all $\phi \in H^2_\mu(\Sigma)$. These operators are highly non-commutative and not essentially normal if the rank $r > 1$. Let

$$\mathcal{T}_\mu(\Sigma) = C^*(T_f :\ f \in \mathcal{C}(\Sigma))$$

denote the **Toeplitz C^*-algebra**

Let μ be a ℓ-hypergeometric measure on $\Sigma \subset \overline{D} \cap V_\ell$ (closed Kepler ball), of type $\binom{x_0,\ldots,x_h}{y_1,\ldots,y_h}$. Consider the stratification

$$\overline{\Sigma} = \bigcup_{0\le i\le \ell} \bigcup_{c\in S_i} c + \Sigma^c, \quad \Sigma^c := \Sigma \cap Z_0^c.$$

For $c \in S_\ell$ define a $H_c(z) := \exp(z|c)$. We regard H_c^n as a **peaking function** on $c + \Sigma^c$.

Theorem

For any tripotent $c \in S_1$ the sequence of probability measures

$$\mu(dz)\frac{|H_c(z)^{2n}|}{\|H_c^n\|_\mu^2}$$

converges to a measure μ^c supported on the boundary component Σ^c which is $(\ell-1)$-hypergeometric of type $\binom{x_0-\frac{a}{2},\ldots,x_h-\frac{a}{2}}{y_1-\frac{a}{2},\ldots,y_h-\frac{a}{2}}$.

Let $\mathcal{T}_\mu(\Sigma)$ be the Toeplitz C^*-algebra on $H^2_\mu(\Sigma)$, associated with an ℓ-hypergeometric measure supported on $\Sigma \subset \overline{D} \cap V_\ell$. Then for every $c \in S_i$, for $0 \le i \le \ell$, there is an irreducible C^*-representation

$$\sigma^c : \mathcal{T}_\mu(\Sigma) \to \mathcal{T}_{\mu^c}(\Sigma^c)$$

acting on $H^2_{\mu^c}(\Sigma^c)$, which is uniquely determined by the property

$$\sigma^c(T_\mu(f)) = T_{\mu^c}(f^c)$$

for all $f \in \mathcal{C}(\Sigma)$, where $f^c \in \mathcal{C}(\Sigma^c)$ is defined by

$$f^c(\zeta) := f(c + \zeta).$$

These representations are pairwise inequivalent and constitute the full spectrum of $\mathcal{T}_\mu(\Sigma)$.

- For the Hardy space $H^2(S)$, we obtain representations on the 'little' Hardy spaces $H^2(S^c)$ on the Shilov boundaries S_0^c of rank $r - i$
- For the weighted Bergman spaces $H_\nu^2(D),\ \nu > p - 1$, we obtain representations on the 'little' weighted Bergman spaces $H^2_{\nu - ia/2}(D_0^c)$ on the boundary components D_0^c of rank $r - i$
- For the boundary orbit spaces $H^2_{\nu_k}(\partial_k D),\ k \leq \ell$ we obtain representations on the 'little' boundary orbit spaces if $i \leq k$ and 'little' weighted Bergman spaces if $k < i \leq \ell$. Thus some boundary orbits 'disappear' in the discrete series when passing to the boundary.

Let (M,ω) be a compact Kähler n-manifold, with a quantizing line bundle $\mathcal{L} \to M$. Consider holomorphic sections $\Gamma(M,\mathcal{L}^\nu)$ of a (high) tensor power $\mathcal{L}^\nu$. **Kodaira (coherent state) embedding**

$$\kappa_\nu : M \to \mathbf{P}(\Gamma(M,\mathcal{L}^\nu)), \; z \mapsto [s_\nu^0(z), .., s_\nu^N(z)]$$

$$D_\nu = \frac{i}{2}\partial\overline{\partial}\log\sum_{i=0}^N |\zeta^i|^2 \text{ Fubini-Study metric on } \mathbf{P}^N$$

$$\kappa_\nu^*(D_\nu) = \nu\omega + \frac{i}{2}\partial\overline{\partial}\log T_\nu$$

$$T_\nu(z) = \sum_{i=0}^{d_\nu} h_z(s_\nu^i(z), s_\nu^i(z)) \text{ Kempf distortion function}$$

TYZ (Tian-Yau-Zelditch) expansion

$$\left|\frac{T_\nu(z)}{\nu^n} - \sum_{j=0}^{N-1}\frac{a_j(z)}{\nu^j}\right| \le \frac{c_N}{\nu^N} \text{ as } \nu \to \infty$$

Theorem: Consider the Fock-type situation, $\alpha = 1$, ℓ arbitrary. There exist polynomials $p_j(x)$, with $p_0 = const$, such that

$$\frac{1}{\nu^n}\mathcal{K}_\nu(\sqrt{x},\sqrt{x}) \approx \frac{e^{\nu\ tr(x)}}{N_c(x)^{n/\ell}} \sum_{j=0}^{n} \frac{p_j(x)}{\nu^j},$$

as $|x| \to +\infty$, uniformly for x in compact subsets of D_c.
Proof: For $Re(\mu) > \frac{d}{r} - 1$ and $Re(\gamma) > \frac{d}{r} - 1$, there is an asymptotic expansion

$${}_1F_1\binom{\mu}{\gamma}(z) \approx \frac{\Gamma_r(\gamma)}{\Gamma_r(\mu)} \frac{e^{tr(z)}}{N(z)^{\gamma-\mu}}\ {}_2F_0\left(\frac{d}{r} - \mu; \gamma - \mu\right)(z^{-1})$$

as $|z| \to +\infty$ while $\frac{z}{|z|}$ stays in a compact subset of D_c. This expansion can be differentiated termwise any number of times. Similar expansion for Kepler ball, based on asymptotics of ${}_2F_1$.

Let Z be an irreducible Jordan triple of rank r. For $1 \le i < j \le r$ put

$$a := \dim Z_{ij},\, b := \dim Z_{0j}$$

These **characteristic multiplicities** are determined by

$$\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b,$$

$$p = 2 + a(r-1) + b.$$

where p is the genus. In the matrix case $Z = \mathbf{C}^{r \times s}$ we have $a = 2,\ b = s - r$, and hence $d/r = s,\ p = r + s$. The **tube type** case corresponds to $b = 0$ or, equivalently, to $\frac{p}{2} = \frac{d}{r}$. Let

$$(u|v) = \frac{1}{p}\, tr_Z\ L(u,v)$$

be the normalized K-invariant inner product.

Segal-Bargmann-Fock space

Consider the **Fischer-Fock inner product**

$$(\phi|\psi)_Z := \frac{1}{\pi^d} \int_Z dz\ e^{-(z|z)}\ \overline{\phi(z)}\ \psi(z) = (\partial_\phi\ \psi)(0)$$

for all $\phi, \psi \in \mathcal{P}(Z)$. Here ∂_ϕ denotes the constant coefficient differential operator on Z induced by ϕ via the inner product $(z|w)$. The Hilbert space completion of $\mathcal{P}(Z)$ is the **Segal-Bargmann-Fock space**

$$H^2(Z) = \{\psi : Z \to \mathbf{C} \text{ holomorphic} : \int_Z \frac{dz}{\pi^d}\ e^{-(z|z)}\, |\psi(z)|^2 < \infty\}$$

of entire functions, which has the reproducing kernel

$$\mathcal{E}(z,w) = e^{(z|w)}.$$

[E]

(M,ω) compact Kähler n-manifold

$\mathcal{L} \to M$ quantizing line bundle

$\mathcal{O}(M,\mathcal{L}^\nu)$ holomorphic sections of power $\mathcal{L}^\nu$

$\sigma_\nu : U \to \mathcal{L}^\nu \setminus 0$ local trivializing section

$\nu\omega(z) = -\frac{i}{2}\partial\overline{\partial}\log h_\nu(\sigma_\nu(z),\sigma_\nu(z))$ Ricci form

$\mathbf{P}(\mathcal{O}(M,\mathcal{L}^\nu))$, $(s_\nu^0, .., s_\nu^{d_\nu})$ ONB of sections

$$\mathbf{P}^{d_\nu} : [Z^0, .., Z^{d_\nu}] \text{ homogeneous coordinates}$$

$$\kappa_\nu : M \ni z \mapsto [s_\nu^0(z), .., s_\nu^{d_\nu}(z)] \in \mathbf{P}(\mathcal{O}(M, \mathcal{L}^\nu)) \text{ coherent state embedding}$$

$$D_\nu = \frac{i}{2}\partial\overline{\partial}\log\sum_{j=0}^{d_\nu}|Z^j|^2 \text{ Fubini-Study metric on } \mathbf{P}^{d_\nu}$$

$$\kappa_\nu^*(D_\nu) = \nu\omega + \frac{i}{2}\partial\overline{\partial}\log T_\nu = \frac{i}{2}\partial\overline{\partial}\log\frac{T_\nu}{h_\nu}$$

$$T_\nu(z) = \sum_{i=0}^{d_\nu} h_z(s_\nu^i(z), s_\nu^i(z)) \text{ distortion function}$$

TYZ(Tian-Yau-Zelditch) expansion

$$\frac{T_\nu(z)}{\nu^n} \approx \sum_{j\geq 0}\frac{a_j(z)}{\nu^j} \text{ as } \nu \to \infty$$

$$\Big|\frac{T_\nu(z)}{\nu^n} - \sum_{j=0}^{N-1}\frac{a_j(z)}{\nu^j}\Big| \leq \frac{c_N}{\nu^N}$$

The **'partial' Hardy space**, denoted by $\mathcal{H}_{S_\ell}$, consists of all holomorphic functions on D_ℓ having boundary values on S_ℓ which are square-integrable for μ_ℓ. The inner product is

$$(\phi|\psi)_{S_\ell} := \int_{S_\ell} du\ \overline{\phi(u)}\ \psi(u).$$

Passing to a Bergman type setting, consider the Kepler ball endowed with the probability measure

$$d\mu_\ell(z) = d\mu_{\ell,\nu}(z) := c_\nu\ \Delta(z,z)^{\nu-p}\ d\lambda_\ell(z).$$

Here $d\lambda_\ell(z)$ is the 'Riemannian' measure on D_ℓ induced by the normalized K-invariant hermitian metric $(z|w)$. The dependence on the spectral parameter ν is tacitly understood. The **'partial' weighted Bergman space** $\mathcal{H}_{D_\ell}$ is defined as the Hilbert space of all holomorphic functions on the Kepler ball D_ℓ which are square-integrable for μ_ℓ. The inner product is

$$(\phi|\psi)_{D_\ell} := \int_{D_\ell} d\mu_\ell(z)\ \overline{\phi(z)}\ \psi(z).$$

Since the underlying measure is K-invariant, the second Riemann extension theorem(a consequence of the normality theorem) implies that a Hilbert space $\mathcal{H}$ of holomorphic functions on the Kepler variety is a Hilbert sum

$$\mathcal{H} = \sum \mathcal{P}_{m_1\ldots,m_\ell,0\ldots,0}(Z)$$

involving only(non-negative) partitions of length $\leq \ell$, since all other components vanish on Z_ℓ.

Proposition: Let $\boldsymbol{m} \in \mathbf{N}_+^\ell$. Then we have for all $p_{\boldsymbol{m}}, q_{\boldsymbol{m}} \in \mathcal{P}_Z^{\boldsymbol{m}}$

$$(p_{\boldsymbol{m}}|q_{\boldsymbol{m}})_{S_\ell} = \frac{(\frac{a}{2}\ell)_{\boldsymbol{m}}}{(\frac{a}{2}r)_{\boldsymbol{m}}} \ \frac{(p_{\boldsymbol{m}}|q_{\boldsymbol{m}})}{(1+\frac{a}{2}(r-1)+b)_{\boldsymbol{m}}},$$

$$(p_{\boldsymbol{m}}|q_{\boldsymbol{m}})_{D_\ell} = \frac{(\frac{a}{2}\ell)_{\boldsymbol{m}}}{(\frac{a}{2}r)_{\boldsymbol{m}}} \ \frac{((1+\frac{a}{2}(2r-\ell-1)+b)_{\boldsymbol{m}}}{(1+\frac{a}{r}-1)+b)_{\boldsymbol{m}}} \ \frac{(p_{\boldsymbol{m}}|q_{\boldsymbol{m}})}{(\nu)_{\boldsymbol{m}}}$$

Choose a base point $c = e_1 + \ldots + e_\ell \in S_\ell$. Every positive density function $\rho(t)$ on symmetric cone Ω_c induces a K-invariant **radial measure**

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_{\Omega_c} dt\ \rho(t) \int_K dk\ f(k\sqrt{t}).$$

For $\ell = 1$ this simplifies to

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_0^\infty dt\ \rho(t) \int_{S_1} du\ f(u\sqrt{t}).$$

Define a **Hilbert space of holomorphic functions**

$$\mathcal{H}_\rho := \{f : \mathring{Z}_\ell \to \mathbf{C} \text{ holomorphic} : \int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ |f(z)|^2 < +\infty\}.$$

Theorem: The Hilbert space $\mathcal{H}_\rho$ has the **reproducing kernel**

$$\mathcal{K}^\rho(z,w) := \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{d_{\boldsymbol{m}}}{\int_{\Omega_c} d\rho(t) E^{\boldsymbol{m}}(t,e)}\ E^{\boldsymbol{m}}(z,w)$$

$$= \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\int_{\Omega_c} d\rho(t) N_{\boldsymbol{m}}(t)}\ \frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}}^c}\ E^{\boldsymbol{m}}(z,w).$$

Every euclidean Jordan algebra X has a **Jordan algebra determinant** $N(x)$ which is a r-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla_x N}{N(x)}.$$

For real or complex matrices, this is the usual determinant. For even anti-symmetric matrices, it is the Pfaffian. In the rank 2 case, $N(x_0, x) = x_0^2 - (x|x)$ is the Lorentz metric.
Let $\partial_N = N(\frac{\partial}{\partial x})$ be the associated constant coefficient differential operator on X. Then $N(x)\partial_N$ is a kind of Euler operator, of order r.
Applying these concepts to the Jordan algebra $Z_c^{(2)}$ of rank ℓ, we define a **universal differential operator**

$$\mathcal{D}_\ell := N^{\frac{a}{2}(\ell-r)} \partial_N^b \ N^{\frac{a}{2}(r-\ell-1)+b+1} \ \left(\partial_N \ N^{\frac{a}{2}} \ \partial_N^{a-1}\right)^{r-\ell} \ N^{\frac{a}{2}(r-\ell+1)-1}$$

of order $\ell((r-\ell)a+b)$ on the symmetric cone D_c. For $\ell = 1$, $N(t) = t$ and $\mathcal{D}_1$ becomes a polynomial differential operator of order $(r-1)a+b$ on the half-line.

Our main result is that the reproducing kernel $\mathcal{K}_\rho(z,w)$ can be expressed in closed form, by

$$\mathcal{K}_\rho(\sqrt{t},\sqrt{t}) = (\mathcal{D}_\ell F_\rho)(t)$$

for all $t \in D_c$, where F_ρ is a special function of **hypergeometric type.** Since hypergeometric functions have good asymptotic expansions(quite difficult for $\ell > 1$), this will lead to the desired TYZ-expansions. Generalizing the 1-dimensional case, Consider first a Fock type situation. For $\alpha > 0$ consider the 'pluri-subharmonic' function $\phi = (z|z)^\alpha$ on Z_ℓ. Let $\omega := \partial\overline{\partial}\phi$ denote the associated Kähler form. Then we have the polar decomposition

$$\int_{\mathring{Z}_\ell} \frac{|\omega^n|}{n!}(z)\ e^{-\nu\phi(z)} f(z) = \int_{D_c} dt\ N_c(t)^{a(r-\ell)+b}\ (t|c)^{n(\alpha-1)}\ e^{-\nu(t|c)^\alpha},$$

where $N_c(t)$ is the rank ℓ determinant function on D_c. Let $\alpha = 1$.
Theorem: The Hilbert space $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t,e) = \mathcal{D}_\ell\ {}_1F_1\binom{d_\ell/\ell}{n/\ell}(\nu t)$$

for the **confluent hypergeometric function**

$${}_1F_1\binom{\sigma}{\tau}(z,w) = \sum_{\boldsymbol{m}} \frac{(\sigma)_{\boldsymbol{m}}}{(\tau)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w).$$

In the bounded setting (Kepler ball) consider the pluri-subharmonic function

$$\phi(z) = \log \Delta(z,z)^{-p} = \log \det B(z,z)^{-1}$$

on the Kepler ball, given by the Bergman kernel. As before, the associated radial measure $e^{-\nu\phi}|\omega^n|/n!$ has a nice polar decomposition, this time involving the unit interval $D_c \cap (c - D_c)$.

Theorem: For the Kepler ball, $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t,e) = \mathcal{D}_\ell \ {}_2F_1\binom{d_\ell/\ell;\nu}{n_\ell/\ell}(t)$$

involving a **Gauss hypergeometric function** ${}_2F_1$ on Ω_c.

Special case, Lie ball $r = 2, \ell = 1$ (Englis et al, Gramchev-Loi)
non-compact Kähler manifold, homogeneous, non-symmetric

$$T^*(\mathbf{S}^n) \setminus \{0\} = \{(x,\xi) : \|x\| = 1,\ (x|\xi) = 0,\ \xi \neq 0\}$$

$\mathbf{C}^{n+1}$ hermitian Jordan triple, rank 2

$\{uv^*w\} = (u|v)w + (w|v)u + (u|\overline{w})\overline{v}$ Jordan triple product

$N(z) = (z|\overline{z})$ Jordan algebra determinant

$T^*(\mathbf{S}^n) \setminus \{0\} = \{z \in \mathbf{C}^{n+1} : N(z) = 0\}$ rank 1 elements

$$\frac{T_\nu(z)}{\nu^n} = 1 + \frac{(n-1)(n-2)}{2|\nu z|} + \sum_{j=2}^{n-2} \frac{2a_j}{|\nu z|^j} + R_\nu(|z|)$$

exponentially small error term

For $\ell = 1$ (minimal Kepler varieties) consider Fock space asymptotic expansion, with α arbitrary

Theorem: Radial measure $d\rho(t) = \alpha e^{-\nu t^\alpha} t^{\alpha(p-1)-1}\, dt$ on half-line. Then, as $\nu \to +\infty$

$$\frac{e^{-\nu(z|z)^\alpha}}{\nu^{p-1}}\mathcal{K}_\nu(z,z) = \sum_{j=0}^{p-2} \frac{b_j}{\nu^j (z|z)^{j\alpha}} + O(e^{-\eta\nu(z|z)^\alpha}),\ \eta > 0$$

Proof: The measure $d\rho$ has moments

$$\alpha \int_0^\infty dt\ e^{-\nu t^\alpha}\ t^{\alpha(p-1)+m-1} = \frac{\Gamma(p-1+\frac{m}{\alpha})}{\nu^{p-1+\frac{m}{\alpha}}}.$$

Therefore $\mathcal{K}_\nu = \mathcal{D}_1 F_\rho$ with

$$F_\rho(t) = \nu^{p-1} \sum_{m=0}^\infty \frac{(\nu^{1/\alpha}t)^m}{\Gamma(p-1+\frac{m}{\alpha})} = \nu^{p-1} E_{1/\alpha,p-1}(\nu^{1/\alpha}t),$$

for the **Mittag-Leffler function**

$$E_{A,B}(t) = \sum_{m=0}^\infty \frac{t^m}{\Gamma(Am+B)}.$$

Theorem:

For $t \in D_c$ define the function

$$F_\rho(t) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\int\limits_{D_c} d\rho \; N_{\boldsymbol{m}}} \; E^{\boldsymbol{m}}(t,c).$$

Every euclidean Jordan algebra X has a **positive cone**

$$\Omega = \{x^2 : \; x \in X \setminus \{0\}\},$$

corresponding to the positive definite matrices $\mathcal{H}_r^+(\mathbf{K})$. The **Jordan algebra determinant** $N : X \to \mathbf{R}$ is a r-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla_x N}{N(x)}.$$

For the rank 2 case,

$$N \begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} = \alpha\delta - (b|b)$$

is the Lorentz metric on $\mathbf{R}^{1,\,1+a}$. The complexification $Z := X^{\mathbf{C}}$ becomes a (complex) Jordan triple(of tube type)

$$\{uv^*w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

For the spin factor we obtain

$$\phi = (z|z)^\alpha,\ \alpha > 0 \text{ pluri-subharmonic function on } Z_\ell$$

$$\omega := \partial\overline{\partial}\phi \text{ Kähler form}$$

$$\int_{\mathring{Z}_\ell} \frac{|\omega^n|}{n!}(z)\ e^{-\nu\phi(z)} f(z) = \int_{D_c} dt\ N_c(t)^{a(r-\ell)+b}\ (t|c)^{n(\alpha-1)}\ e^{-\nu(t|c)^\alpha}.$$

Theorem: $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has kernel $\mathcal{K}_\nu(\sqrt{t},\sqrt{t}) = \mathcal{D}_\ell F_\nu(t)$

$$F_\nu(t) = const \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\Gamma_{D_c}(\boldsymbol{m}+\frac{n}{\ell})}\ \frac{\Gamma(n+|\boldsymbol{m}|)}{\Gamma(n+\frac{|\boldsymbol{m}|}{\alpha})}\ E^{\boldsymbol{m}}(\nu^{1/\alpha}t,c)$$

confluent hypergeometric function

$${}_1F_1(z,w) = \sum_{\boldsymbol{m}} \frac{(\sigma)_{\boldsymbol{m}}}{(\tau)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w)$$

For $\alpha = 1$

$$F_\nu(t) = {}_1F_1\binom{d_\ell/\ell}{n/\ell}(\nu t)$$

Theorem: $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has kernel $\mathcal{K}_\nu(\sqrt{t},\sqrt{t}) = \mathcal{D}_\ell F_\nu(t)$

$$F_\nu(t) = const \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\Gamma_{D_c}(\boldsymbol{m}+\frac{n}{\ell})}\ \frac{\Gamma(n+|\boldsymbol{m}|)}{\Gamma(n+\frac{|\boldsymbol{m}|}{\alpha})}\ E^{\boldsymbol{m}}(\nu^{1/\alpha}t,c)$$

***

The invariant probability measure dg on G^+ satifies

$$\int_{G^+} dg\, f(g(0)) = \frac{\Gamma_\Omega(p)}{\Gamma_\Omega(p-\frac{d}{r})} \int_Z \frac{d\overline{w}dw}{(2\pi i)^d}\ \Delta_{w,-w}^{-p}\, f(w)$$

for all $f \in L^\infty(Z)$. Here dw denotes the Lebesgue measure for the inner product (.), and the normalization constant is computed in terms of the Gindikin Γ_Ω-function .

Spectral Analysis Since $Z^+ = G^+/K$ is a compact symmetric space, the space $L^2(Z^+)$, for the normalized Haar measure (1.), has a multiplicity-free "Peter-Weyl" decomposition into irreducible G^+-submodules. This "Peter-Weyl" decomposition can be described in a uniform manner:

$$L^2(Z^+) = \sum_{\boldsymbol{m}} L^2_{\boldsymbol{m}}(Z^+) \tag{0.1}$$

where $\boldsymbol{m}$ runs over all partitions of length r, and $L^2_{\boldsymbol{m}}(Z^+)$ is the (finite-dimensional) G^+-submodule generated by the *spherical function* $\Phi^+_{\boldsymbol{m}}$.

example

In the rank 1 case $Z^+ = \mathbf{CP}^1 \approx \mathbf{S}^2$, $L^2_{\boldsymbol{m}}(Z^+)$ is spanned by the functions

$$z \mapsto \left[\frac{\alpha z + \beta \overline{z} + \gamma(z\overline{z} - 1)}{1 + |z|^2} \right]$$

where $\alpha, \beta, \gamma \in \mathbf{C}$ satisfy $\alpha, \beta + 4\gamma^2 = 0$.The corresponding spherical function is

$$\Phi^+_m(z) = C_m \left(\frac{1 - |z|^2}{1 + |z|^2} \right)^m ,$$

where C_m is the m-th Gegenbauer polynomial.

Invariant volume forms and measures

$\mathring{Z}_\ell$ is homogeneous under $K^{\mathbf{C}}$ or a suitable transitive subgroup. When does an invariant holomorphic volume form or invariant measure exist? Only for domains of tube type $b = 0$.

- **Theorem:** Let Z be of tube type and $0 < \ell < r$. Then there exists an invariant holomorphic n-form Ω on $\mathring{Z}_\ell$ if and only if $p - a\ell = 2 + a(r - \ell - 1)$ is even.
- Among all tube type domains, $p - a\ell$ is odd only for the symmetric matrices $Z = \mathbf{C}^{r\times r}_{sym}$ with $r - \ell$ even.
- **Theorem:** An invariant measure μ on $\mathring{Z}_\ell$ exists in all cases (for $b = 0$), and has polar decomposition

$$\int_{\mathring{Z}_\ell} d\mu(z)\ f(z) = const \int_{D_c} \frac{dt}{N_c(t)^{d_\ell/\ell}}\ N_c(t)^{ar/2} \int_K dk\ f(k\sqrt{t}).$$