On Isometries Satisfying Deformed Commutation Relations

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The C^*-algebra $\mathcal{E}_{1,n}^q$

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$$s_i^* s_j = 0,\ i \neq j, \quad t^* s_j = q s_j t^*.$$

The Cuntz-Toeplitz algebra $\mathcal{O}_n^0$

Definition (Cuntz, 1977)

The Cuntz-Toepltiz C^*-algebra $\mathcal{O}_n^0$ is the universal C^*-algebra generated by family of isometries $v_1, \dots, v_n$ with orthogonal ranges. I.e.

$$v_i^* v_i = \mathbf{1}, \quad v_i^* v_j = 0, \quad i \neq j.$$

The case $|q| < 1$, $\varphi\colon \mathcal{E}_{1,n}^{0} \to \mathcal{E}_{1,n}^{q}$

Theorem

Let $|q| < 1$. Put $\hat{s}_j = (\mathbf{1} - tt^*)s_j(\mathbf{1} - |q|^2 tt^*)^{-1/2}$, $j = \overline{1,n}$. Then the family $\{t, \hat{s}_j\}_{j=1}^{n}$ generates $\mathcal{E}_{1,n}^{q}$, and

$$t^*t = \mathbf{1}, \quad t^*\hat{s}_j = 0, \quad \hat{s}_j^*\hat{s}_i = \delta_{ij}\mathbf{1}, \qquad i,j = \overline{1,n}.$$

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Corollary

Denote by v_i, $i = \overline{1, n+1}$, the isometries generating $\mathcal{E}^0_{1,n} = \mathcal{O}^0_{n+1}$. Then Theorem 2 implies that the correspondence

$$v_1 \mapsto t, \quad v_{1+j} \mapsto \hat{s}_j, \quad j = \overline{1, n},$$

extends uniquely to a surjective homomorphism $\varphi\colon \mathcal{E}^0_{1,n} \to \mathcal{E}^q_{1,n}$.

The case $|q| < 1$, $\psi\colon \mathcal{E}^q_{1,n} \to \mathcal{E}^0_{1,n}$

Let us construct the inverse homomorphism. $\psi\colon \mathcal{E}^q_{n,m} \to \mathcal{E}^0_{n,m}$.

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Theorem

Let v_i, $i = \overline{1, n+1}$, be the isometries generating $\mathcal{E}^0_{1,n}$. Put

$$\tilde{w}_j = v_{1+j}(\mathbf{1} - |q|^2 v_1 v_1^*)^{1/2} \quad \text{and} \quad w_j = \sum_{k=0} q^k v_1^k \tilde{w}_r (v_1^k)^*.$$

Then

$$w_j^* w_i = \delta_{ji}\mathbf{1}, \quad v_1^* w_j = q w_j v_1^*, \quad i, j = \overline{1, n}.$$

Moreover, the family $\{v_1, w_j\}_{j=1}^n$ generates $\mathcal{E}^0_{1,n}$.

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Moreover, the family $\{v_1, w_j\}_{j=1}^n$ generates $\mathcal{E}^0_{1,n}$.

Corollary

The statement of Theorem 4 and the universal property of $\mathcal{E}^q_{1,n}$ imply the existence of a surjective homomorphism $\psi\colon \mathcal{E}^q_{1,n} \to \mathcal{E}^0_{1,n}$ defined by

The case $|q| < 1$, $\mathcal{E}^q_{1,n} \simeq \mathcal{E}^0_{1,n}$

Theorem

For any $q \in \mathbb{C}$, $|q| < 1$, one has an isomorphism $\mathcal{E}^q_{1,n} \simeq \mathcal{E}^0_{1,n}$.

The case $|q| = 1$

In this case the generating relations imply the relation of the form

$$s_j t = q t s_j, \quad j = 1, \dots, n.$$

Theorem

The C^*-algebra $\mathcal{E}_{1,n}^q$ is nuclear.

Definition

The Fock representation, π_F^q, of $\mathcal{E}_{1,n}^q$, is the unique up to unitary equivalence irreducible $*$-representation having the vacuum vector Ω, $||\Omega|| = 1$, such that

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$$\pi_F^q(s_j^*)\Omega = 0, \quad \pi_F^q(t^*)\Omega = 0, \quad j = \overline{1, n}.$$

The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$

Let

$$S\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+), \quad Se_n = e_{n+1},$$

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where e_n, $n \in \mathbb{Z}_+$, are vectors of standard orthonormal basis.

The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$

Denote by $\pi_{F,n}$ the Fock representation of $\mathcal{O}_n^{(0)} \subset \mathcal{E}^q_{1,n}$ acting on the space

$$\mathcal{F}_n = \mathcal{T}(\mathcal{H}_n) = \mathbb{C}\Omega_n \oplus \bigoplus_{d=1}^{\infty} \mathcal{H}_n^{\otimes d}, \quad \mathcal{H}_n = \mathbb{C}^n,$$

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$$\pi_{F,n}(s_j)\Omega_n = e_j, \quad \pi_{F,n}(s_j)e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d} = e_j \otimes e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d},$$
$$\pi_{F,n}(s_j^*)\Omega = 0, \quad \pi_{F,n}(s_j^*)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} = \delta_{ji_1} e_{i_2} \otimes \cdots \otimes e_{i_d}.$$

The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$

Let $d_n(q)\colon \mathcal{F}_n \to \mathcal{F}_n$ the unitary operator, such that

$$d_n(q)\colon \mathcal{F}_n \to \mathcal{F}_n, \quad d_n(q)\Omega_n = \Omega_n, \quad d_n(q)X = q^d X,\ X \in \mathcal{H}_n^{\otimes d}.$$

The Fock representation of $\mathcal{E}_{1,n}^q$, $|q|=1$

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Theorem

The Fock representation of $\mathcal{E}_{1,n}^q$ exists. Up to a unitary equivalence, the Fock space $\mathcal{F}^q = l_2(\mathbb{Z}_+) \otimes \mathcal{F}_n$ and

$$\pi_F^q(t) = S \otimes \mathbf{1}_{\mathcal{F}_n},$$
$$\pi_F^q(s_j) = d_1(q) \otimes \pi_{F,n}(s_j), \quad j = 1, \ldots, d.$$

The Fock representation of $\mathcal{E}_{1,n}^q$, $|q| = 1$

Remark

Notice that π_F^q is faithful on $*$-subalgebra $E_{1,n}^q \subset \mathcal{E}_{1,n}^q$ generated by t and s_j, $j = \overline{1, n}$.

The Fock representation of $\mathcal{E}_{1,n}^{q}$, $|q| = 1$

Remark

Below we present two another forms of generators of $\mathcal{E}_{1,n}^{q}$ in the Fock representation,

$$\pi_F^q(s_j) = \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \pi_{F,n}(s_j), \quad j = \overline{1, n},$$
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$$\pi^q_F(t) = S \otimes d_n(q^{-1}),$$

or

$$\pi^q_F(s_j) = d(q^{1/2}) \otimes \pi_{F,n}(s_j), \quad j = \overline{1, n},$$
$$\pi^q_F(t) = S \otimes d_n(q^{-1/2}).$$

The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$

Next we show that π^q_F is faithful representation of $\mathcal{E}^q_{1,n}$. To this end we consider the action α of $\mathbb{T}^2$ on $\mathcal{E}^q_{n,m}$,

$$\alpha_{\varphi_1,\varphi_2}(s_j) = e^{2\pi i \varphi_1} s_j, \quad \alpha_{\varphi_1,\varphi_2}(t) = e^{2\pi i \varphi_2} t.$$

In the following we denote by Λ_n the set of all words in alphabet $\{1, 2, \ldots, n\}$ including the empty word.

Proposition

The fixed point C^*-subalgebra $(\mathcal{E}^q_{1,n})^\alpha \subset \mathcal{E}^q_{1,n}$ with respect to α is an AF-algebra and the restriction of π^q_F to $(\mathcal{E}^q_{1,n})^\alpha$ is faithful.

The Fock representation of $\mathcal{E}^q_{1,n}$, $|q| = 1$

Corollary

The Fock representation of $\mathcal{E}^q_{1,n}$ *is faithful.*

The Fock representation of $\mathcal{E}_{1,n}^q$, $|q| = 1$

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The Fock representation of $\mathcal{E}_{1,n}^q$ is faithful.

Proof.

It is easy to see that π_F^q is equivariant homomorphism between C^*-algebras $\mathcal{E}_{1,n}^q$ and $\pi_F^q(\mathcal{E}_{1,n}^q)$. It remains to notice that equivariant homomorphism between C^*-algebras with group action is faithful iff it is faithful on fixed point subalgebras. □

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Denote

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Theorem[Generalised Wold decomposition] Let $\pi\colon \mathcal{E}_{1,n}^{q} \to \mathbb{B}(\mathcal{H})$ be a $*$-representation. Then

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

each $\mathcal{H}_j$, $j = 1, 2, 3, 4$, is invariant with respect to π.

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For $\pi_j = \pi \restriction_{\mathcal{H}_j}$ one has

- $\mathcal{H}_1 = \mathcal{F} \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$, and $\pi_1 = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$;
- $\mathcal{H}_2 = l_2(\mathbb{Z}_+) \otimes \mathcal{G}_2$, $\pi_2(\mathbf{1} - Q) = 0$, $\pi_2(\mathbf{1} - P) \neq 0$,

  $$\pi_1(t) = S \otimes \mathbf{1},\ \pi_1(s_j) = d_1(q) \otimes \tilde{\pi}_1(s_j),\ j = \overline{1, n},$$

  where operators $\tilde{\pi}_1(s_j)$ determine representation of $\mathcal{O}_n$ on $\mathcal{G}_2$;
- $\mathcal{H}_3 = \mathcal{F}_n \otimes \mathcal{G}_3$, $\pi_3(\mathbf{1} - P) = 0$, $\pi_3(\mathbf{1} - Q) \neq 0$,

  $$\pi(s_j) = \pi_{F,n}(s_j) \otimes 1, \quad \pi(t) = d_n(q^{-1}) \otimes U,$$

  where U is unitary on $\mathcal{G}_3$;
- $\pi_4(\mathbf{1} - Q) = 0$, $\pi_4(\mathbf{1} - P) = 0$;

where any of $\mathcal{H}_j$, $j = 1, 2, 3, 4$, could be zero.

Representations of $\mathcal{E}_{1,n}^q$

Here we describe exactly the classes of unitary equivalence of representations of $\mathcal{E}_{1,n}^q$ such that the unitary part in the Wold decomposition of isometry corresponding to some of s_j, $j = \overline{1,n}$, is non-zero.

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For $\pi\colon \mathcal{E}_{1,n}^q \to B(\mathcal{H})$:

$$\pi(t) := T, \quad \pi(s_j) := S_j,\ j = \overline{1,n}.$$

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Put

$$\Lambda_n^j = \{\lambda = (\lambda_1, \cdots, \lambda_k) \mid 1 \le \lambda_j \le n,\ \lambda_k \ne j,\ k \in \mathbb{N}\}.$$

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We also denote by

$$S_\lambda := S_{\lambda_1} \cdots S_{\lambda_k}.$$

Proposition

Let $\pi\colon \mathcal{E}^q_{1,n} \to B(\mathcal{H})$ *be representation such that the unitary part of* $S_j = \pi(s_j)$ *is non-zero. Denote by* $\mathcal{H}_{u,j} \subset \mathcal{H}$ *the largest subspace invariant with respect to* S_j, S_j^*, *such that the restriction of* S_j *is unitary. Then* $\mathcal{H}_{u,j}$ *is invariant with respect to* T, T^*,

$$S_k^*(\mathcal{H}_{u,j}) = \{0\}, \quad S_\lambda(\mathcal{H}_{u,j}) \perp S_\mu(\mathcal{H}_{u,j}),\ k \neq j,\ \lambda \neq \mu,\ \lambda, \mu \in \Lambda_n^j.$$

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Let $\widetilde{T}$ and V_j be restrictions of T and S_j resp. to $\mathcal{H}_{u,j}$.

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Proposition

Let $\pi\colon \mathcal{E}_{1,n}^q \to B(\mathcal{H})$ be such that $\pi(s_j)$ has non-trivial unitary part, and $\mathcal{H} = \widetilde{\mathcal{H}}$ introduced above. Then π is irreducible iff the family $\{\widetilde{T}, \widetilde{T}^*, V_j\}$ is irreducible on $\mathcal{H}_{u,j}$.

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Proposition

Let $\pi_k\colon \mathcal{E}_{1,n}^q \to \mathcal{H}_k$, $k = 1, 2$, be irreducible representations such that $\pi_k(s_j)$ have non-trivial unitary part. Then π_1 is unitary equivalent to π_2 iff the corresponding families $\mathfrak{F}_k = \{\widetilde{T}_k, \widetilde{T}_k^*, V_{j,k}\}$, $k = 1, 2$, acting on $\mathcal{H}_{u,j}^{(k)}$ are unitary equivalent.

Combining results above we get the following

Combining results above we get the following **Theorem** Let π be irreducible representation of $\mathcal{E}_{1,n}^{q}$ on the space $\mathcal{H}$, such that for a fixed $j = \overline{1, n}$ the isometry $S_j = \pi(s_j)$ has non-trivial unitary part in its Wold decomposition.

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Then π is determined uniquely, up to the unitary equivalence, by irreducible family of operators $\{\widetilde{T},\widetilde{T}^*,V_j\}$ on Hilbert space $\mathcal{G}$, satisfying the relations

$$\widetilde{T}^*\widetilde{T}=\mathbf{1},\ V_j^*V_j=V_jV_j^*=\mathbf{1},\ \widetilde{T}^*V_j=qV_j\widetilde{T}^*,\ V_j\widetilde{T}=q\widetilde{T}V_j.$$

Namely, let $f^{(i)}$, $i \in \mathcal{I}$ be orthonormal basis of $S_k\mathcal{G}$. Then the orthonormal basis of $\mathcal{H}$ has the form

$$
\begin{aligned}
f^{(i)} \otimes f_\emptyset,\ f^{(i)} \otimes f_\lambda, \quad i \in \mathcal{I},\ \lambda \in \Lambda_n^j,\\
S_j f^{(i)} \otimes f_\emptyset = (V_j f^{(i)}) \otimes f_\emptyset\\
T f^{(i)} \otimes f_\emptyset = (\widetilde{T} f^{(i)}) \otimes f_\emptyset,\ i \in \mathcal{I},\\
S_k f^{(i)} \otimes f_\emptyset = f^{(i)} \otimes f_{(k)},\ k \neq j,\\
S_k f^{(i)} \otimes f_\lambda = f^{(i)} \otimes f_{\sigma_k(\lambda)},\ k = \overline{1,n},\ i \in \mathcal{I}\\
S_k^* f^{(i)} \otimes f_\emptyset = 0,\ k \neq j, \quad S_k^*\\
f^{(i)} \otimes f_\lambda = \delta_{k\lambda_1} f^{(i)} \otimes f_{\sigma(\lambda)},\ k = \overline{1,n},\ i \in \mathcal{I},\\
T f^{(i)} \otimes f_\lambda = q^{-|\lambda|}(\widetilde{T} f^{(i)}) \otimes f_\lambda,\\
T^* f^{(i)} \otimes f_\lambda = q^{|\lambda|}(\widetilde{T}^* f^{(i)}) \otimes f_\lambda,\ \lambda \in \Lambda_n^j,
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$\sigma_k(\lambda) = (k\lambda)$, $\sigma((\lambda_1)) = \emptyset$, $\sigma((\lambda_1, \ldots, \lambda_m)) = (\lambda_2, \ldots, \lambda_m)$.

Commutative model for representations

The generalized Wold decomposition for representations of $\mathcal{E}_{1,n}^q$, implies in particular, that any irreducible representation of $\mathcal{E}_{1,n}^q$ contains only one component $\mathcal{H}_i$, $j = 1, 2, 3, 4$.

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In $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ all representations are described either explicitly or in terms of representations of $\mathcal{O}_n$. Here, we give a general form of representations in $\mathcal{H}_4$ (commutative model with respect to a commutative subalgebra).

Commutative model for representations

The defining relations in $\mathcal{E}^q_{1,n}$ imply that in $\mathcal{H} = \mathcal{H}_4$, the operator T is unitary, and the operators $P_\lambda = S_\lambda S_\lambda^*$ form a commutative family of projections, all of them commute with T, and

$$\sum_{|\lambda|=k} P_\lambda = I, \quad k = 1, 2, \ldots .$$

Commutative model for representations

Write the joint spectral decomposition of the commuting family $(T, P_\lambda \mid \lambda \in \{1, \ldots, n\}_0^\infty)$ as

$$\mathbb{T} \times \{1, \ldots, n\}^\infty \ni (t, \lambda) = (t, \lambda_1, \lambda_2, \ldots),$$

$$T = \int_{\mathbb{T}\times} \{1, \ldots, n\}^\infty t \, dE(t, \lambda),$$

$$P_\lambda = \int_{\mathbb{T}\times\{1,\ldots,n\}^\infty} \chi_{\lambda\times\{1,\ldots,n\}^\infty}(\lambda) \, d\mu(t, \lambda) = \\ = E(\mathbb{T} \times \mu \times \{1, \ldots, n\}^\infty).$$

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For each $j = 1, \ldots, n$, we have

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Applying general commutative model formalism we get
Theorem For any representation of $\mathcal{E}_{1,n}^q$ in the component $\mathcal{H}_4$ of the generalized Wold decomposition, the following holds. The space $\mathcal{H} = \mathcal{H}_4$ decomposes into a direct integral

$$\mathcal{H} = \int_{\mathbb{T} \times \{1,\dots,n\}^\infty}^{\oplus} \mathcal{H}_{t,\lambda} \, d\mu(t, \lambda) \quad (t, \lambda) = (t, \lambda_1, \lambda_2, \dots),$$

and the operators act by the following formula

$$(Uf)(t,\lambda) = tf(t,\lambda),$$

$$(S_j f)(t,\lambda) = \delta_{j,\lambda_1} U_j(\bar{q}t,\sigma(\lambda))\Big(\frac{d\delta_j(\lambda_1)\otimes\mu(\bar{q}t,\sigma(\lambda))}{d\mu(t,\lambda)}\Big)^{1/2} f(\bar{q}t,\sigma(\lambda)),$$

$$(S_j^* f)(t,\lambda) = U_j^*(qt,\lambda)\Big(\frac{d\mu(qt,\sigma_j(\lambda))}{d\mu(t,\lambda)}\Big)^{1/2} f(qt,\sigma_j(\lambda)).$$

- μ is a probability measure defined on the cylinder σ-algebra, quasi-invariant with respect to the transformations

$$\mu(t, \lambda) \mapsto (qt, \sigma_j(\lambda)),\ j = 1, \ldots, n;$$

- $\mathcal{H}_{t,\lambda}$ is a measurable field of Hilbert spaces such that

$$\dim H_{t,\lambda} = \dim H_{qt,\sigma_j(\lambda)}, \mu - \text{a.e.};$$

- $U_j(t, \lambda)$, $j = 1, \ldots, n$, are measurable unitary operator-valued functions.

Conversely, any

- quasi-invariant measure μ,
- measurable field $\mathcal{H}_{t,\lambda}$ with $\dim H_{t,\lambda} = \dim H_{qt,\sigma_j(\lambda)}$ μ-a.e.,
- a collection of measurable unitary $U_j(t, \lambda)$, $j = 1, \ldots, n$,

give rise to a representation of $\mathcal{E}_{1,n}^{q}$ in the component $\mathcal{H}_4$ of the generalized Wold decomposition.

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- Put μ to be the atomic measure uniformly distributed over the points of O_{t_0,λ_0} and

$$H_{t,\lambda} = \mathbb{C}, \quad U_j(t, \lambda) = 1,\ j = 1, \dots, n.$$

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Remark

Notice that representations corresponding to the points $(t, (j, j, \ldots))$, $j = 1, \ldots, n$, fall in, but does not cover the class of representations with non-tirivial unitary part.

THANK YOU!