Shimura operators and Okounkov interpolation polynomials

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Goal

Answer a question of Shimura on the positivity of invariant differential operators.

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Outline

- Introduction/Motivation: Binomial coefficients as eigenfunctions. Shimura operator on the hyperbolic plane.
- Hermitian symmetric domains and Shimura invariant differential operators. Question of Shimura Positivity
- Results on the positivity

Binomial coefficients as eigenvalues of differential operators. Vanishing properties

Binomial coefficient $m!\binom{n}{m}$ is a polynomial in n (for fixed m),

$$P_m : n \mapsto m!\binom{n}{m} = n(n-1)\cdots(n-m+1) := (n)^{-}_{m}$$

Two equivalent descriptions of $P_m(\lambda) = m!\binom{\lambda}{m}$:
(a) Interpolation (monomial) polynomials at $\lambda = 0, \cdots, m-1$.
(b) Eigenvalues of the differential operators $\frac{d^m}{dt^m}$ on the eigenfunction t^λ, $t \in (0, \infty)$,

$$\frac{d^m}{dt^m} t^\lambda = P_m(\lambda) t^\lambda$$

Shimura operators on the unit disc. Prototype

- Unit disc $H = \{z = x + iy, y > 0\}$, equipped with the hyperbolic metric
$$\frac{|dz|^2}{y^2}$$
Symmetry group $G = SU(1,1)$, the Möbius group of fractional transformtions $z \to (az + b)(cz + d)^{-1}$.

- L^2-space $L^2(H, \mathcal{K}^l)$ of sections $f = f(z)(dz)^l$ of $\mathcal{K}^L$, $\mathcal{K} = T^*$,

$$\int_H |f|_z^2 y^{-2} dxdy = \int_H |f(z)|^2 y^{2k} y^{-2} dxdy < \infty$$

- Eichler operator/Chern connection

$$D = y^{-2l} \frac{\partial}{\partial z} y^{2l} : L^2(H, \mathcal{K}^l) \to L^2(H, \mathcal{K}^{l+1}).$$

Shimura operators

- $$L_m := D^m(D^*)^m, \quad M_m := (D^*)^m D^m$$
- Eigenfunctions (Harish-Chandra e-functions or spherical functions) (for the trivial line bundle $l = 0$)
$$e_\lambda(z) = y^{\frac{1+\lambda}{2}}$$
- Eigenvalue
$$L_m e_\lambda = \frac{1}{4}(-1)^m P_m(\lambda) e_\lambda$$
$$P_m(\lambda) := \prod_{j=0}^{m-1} ((1+2j)^2 - \lambda^2)$$

Sets of positivity and unitarity

- Two interesting sets:

$$U_p := \{\lambda \in \mathbb{C}; \text{all eigenvalues of } L_m \text{ are positive}\}$$

$$U_{uni} = \{\lambda \in \mathbb{C}; e_\lambda \quad \text{generates a unitary representation of } G\}$$

- Generally

$$U_{uni} \subset U_p$$

- Claim

$$U_p = U_{uni} = i\mathbb{R} \cup [-1, 1]$$

  $i\mathbb{R}$ corresponding to principal series representation, and $[-1, 1]$ complementary series

Even interpolation polynomials as eigenvalues of the Shimura operator L_m

Fix $\rho \in \mathbb{R}$. Look for even monomial polynomials $P_m(\lambda)$ of degree $2m$ (i.e. polynomials of λ^2) with zeros

$$\rho, \rho + 1, \cdots \rho + m - 1$$

$$P_m(\lambda) = \prod_{j=0}^{m-1} (\lambda^2 - (\rho + j)^2)$$

Hermitian symmetric spaces G/K

$$G = U(n, m), Sp(n, \mathbb{R}), \ldots \; K = U(n) \times U(m), U(n), \ldots$$

Cartan and Harish-Chandra decompositions

- Choose reference point $o = eK \in M$, $T_0(M)$ identified with $\mathfrak{p}$ in the Cartan deco

$$\mathfrak{g} = Lie(G) = \mathfrak{k} + \mathfrak{p}$$

- Complexification

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{+} + \mathfrak{p}^{-}$$

  $\mathfrak{p}^{+}$ identified with holomorphic tangent space $T_0^{(1,0)}(M)$

Hua-Schmid-Kostant decomposition

- $\mathfrak{p}^+$ irreducible representation of K
- Symmetric tensor space $S_m(\mathfrak{p})$ (=polynomial algebra on $\mathfrak{p}^-$) decomposes under K

$$S_m = \sum_{\underline{\mathbf{m}}: m_1 + \cdots + m_n = r} S_{\underline{\mathbf{m}}}$$

Shimura operators

- $\{p_\alpha\}$ basis of $S_{\underline{\mathbf{m}}}(\mathfrak{p}^+)$, $\{q_\alpha\}$ dual basis in $S_{\underline{\mathbf{m}}}(\mathfrak{p}^-)$, all viewed as differential operaotors on $C^\infty(G)$ by differentiation from right.
- The operators

$$L_{\underline{\mathbf{m}}} = \sum_\alpha q_\alpha p_\alpha$$

decend to operators on $C^\infty(M)$

Properties of $L_{\underline{\mathbf{m}}}$

- Linear basis for the space of all G-invariant differential operators
- $L_{\underline{\mathbf{m}}}$ for

$$\underline{\mathbf{m}} = 1^j = (1, \cdots, 1, 0, \cdots, 0), \quad j = 1, ..., n$$

  algebraic basis.
- $L_{\underline{\mathbf{m}}}$ are commuting positive self-adjoint operators on $L^2(M)$ (more precisely, on some appropriate defining domain)

Eigenfunctions. Eigenvalue polynomials

Harish-Chandra *e*-function are eigenfunctions:

$$L_{\underline{\mathbf{m}}} e_{\underline{\mathbf{s}}} = L_{\underline{\mathbf{m}}}(\underline{\mathbf{s}}) e_{\underline{\mathbf{s}}}$$

Questions of Shimura

Determine the following sets

- $$U_p := \{\underline{\mathbf{s}} \in \mathbb{C}^n; L_{\underline{\mathbf{m}}}(\underline{\mathbf{s}}) \geq 0 \, \forall \underline{\mathbf{m}}\}$$
- Positivity set for generators
$$U_g := \{\underline{\mathbf{s}} \in \mathbb{C}^n; L_{\underline{\mathbf{m}}}(\underline{\mathbf{s}}) \geq 0 \, \forall \underline{\mathbf{m}} = 1^j, j = 1, \cdots, n$$
- Unitarity set (main goal of representation theory of semisimple Lie groups)
$$U_{uni} := \{\underline{\mathbf{s}} \in \mathbb{C}^n; \phi_{\underline{\mathbf{s}}}(z) = \int_K e_{\underline{\mathbf{s}}}(kz) \quad \text{is positive definite}\}$$

Motivations. Examples

- U_{uni} parametrize unitary spherical representations of G. Still an unknown (and central) question
- $$U_{uni} \subset U_p \subset U_g$$
- For rank one domains, i.e. the unit ball in $\mathbb{C}^n$ as symmetric space of $SU(n,1)$, the three sets agree,

$$U_{uni} = U_p = U_g = i\mathbb{R} \cup (-n, n)$$

- Shimura found examples for rank two domains

$$U_p \subsetneq U_g$$

Classical examples: Cayley-Capelli identities and reformulation.

Consider the polynomial

$$\det(x), \quad x \in M_{n,n}(\mathbb{C})$$

Dualizing

$$\det(\partial)$$

constant coefficient invariant differential operator acting on polynomials (say) on $M_{n,n}(\mathbb{C})$

Cayley-Capelli identity (Cayley, 1821-1895; Capelli, 1855-1910; Maass, 1911-1992; Siegel, 1896-1981; Gårding, 1919-2014; Shimura, 1930-...)

$$\det(\partial)\det(x)^{\mu} = \mu(\mu+1)\cdots(\mu+n-1)\det(x)^{\mu-1}$$

Rewriting: Define

$$C = \det(x)\det(\partial)$$

$$\boxed{C\det(x)^{\mu} = \mu(\mu+1)\cdots(\mu+n-1)\det(x)^{\mu}}$$

Abstract formulation.

- $\det(x)\det(\partial)$: $GL(n, \mathbb{C})$-invariant differential operators (acting on polynomial space).
- Eigenvalues: Harish-Chandra homomorphism

Okounkov interpolation polynomials root systems of Type BC

Root system of type BC of rank n in $\mathfrak{a} = \mathbb{R}^n$

$$\{\pm 2e_j; \pm e_j; \pm e_i \pm e_j\}$$

with multiplicity $1, 2b, a$.
Half sum of positive roots ρ.
W=Weyl group = Symmetric group $S_n \times \mathbb{Z}_2^n$.
Weight lattices

$$\sum_j \lambda_j e_j = (\lambda_1, \cdots, \lambda_n); \lambda_1 \geq \lambda_2 \cdots \lambda_n \geq 0$$

with partial ordering

$$\mu \subset \lambda : \mu_j \leq \lambda_j \forall j$$

Symmetric power sum,

$$m_\lambda(x) = \sum_{\mu \in S_n \lambda} x^{2\mu} = x_1^{2\lambda_1} \cdots x_n^{2\lambda_n} + \cdots .$$

Theorem

(Okounkov 1998) For each λ there exists a unique interpolation polynomial $P_\lambda(x) = P_\lambda^{ip}$ of $x, \cdots, x_n$ with leading term m_λ (in the above ordering) such that

$$P_\lambda(x+\rho)$$

is a symmetric polynomial in $x_1^2, \cdots, x_n^2$, and

$$P_\lambda(\mu) = 0$$

unless $\lambda \subset \mu$.

Eigenvalues of Shimura generators

Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, Cartan subspace $\mathfrak{a} = \mathbb{R}^n \subset \mathfrak{p}$.
Harish-Chandra homomorphsim

$$\eta : \mathbf{D}\left(G/K\right) \to S\left(\mathfrak{a}\right)^W \approx \mathcal{P}\left(\mathfrak{a}^*\right)^W . \tag{1}$$

Eigenvalues of Shimura operators

Theorem

(Sahi-Z.) The eigenvalues of the Shimura operators are given by the Okounkov polynomials for root systems of type BC, i.e.,

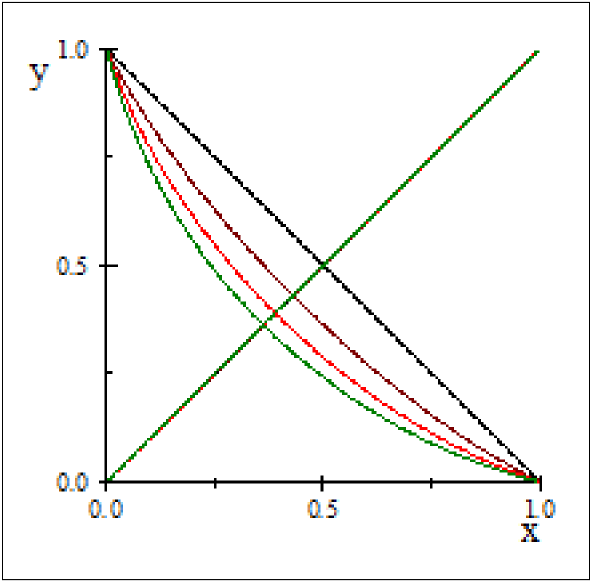
$$\eta(L_\lambda)(x) = P_\lambda(x) \tag{2}$$

Example. Eigenvalues of Shimura operators

The eigenvalues of the operator $\mathcal{L}_{l1^r}$ is

$$c\prod_{j=1}^{n}\prod_{k=0}^{l-1}(x_j^2+(\frac{1+b}{2}+k)^2)$$

The set of positivity. Rank two examples



Set of positivity of eigenvalues vs spherical unitary dual

Further Questions

Shimura operators on q-bounded symmetric domains and their Harish-Chandra homomorphisms?