Numerical Linear Algebra: main notations

Identity matrix

The identity matrix or unit matrix of size n is the $n \times n$ square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by I_n, or simply by I.

$$I_1 = \begin{bmatrix} 1 \end{bmatrix}, \; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \cdots, \quad I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

When A has size $m \times n$, it is a property of matrix multiplication that $I_m A = A I_n = A$.
Using the notation that is sometimes used to concisely describe diagonal matrices, we can write:
$I_n = \mathrm{diag}(1, 1, ..., 1)$.
It can also be written using the Kronecker delta notation:
$(I_n)_{ij} = \delta_{ij}$.

Triangular matrix

- A square matrix is called lower triangular if all the entries above the main diagonal are zero.

$$L = \begin{bmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \dots & l_{n,n-1} & l_{n,n} \end{bmatrix}$$

- A square matrix is called upper triangular if all the entries below the main diagonal are zero.

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \dots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{bmatrix}$$

Triangular matrix

- A triangular matrix is one that is either lower triangular or upper triangular.
- A matrix that is both upper and lower triangular is a diagonal matrix.

$$D_n = \begin{bmatrix} d_{1,1} & 0 & \cdots & 0 \\ 0 & d_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n,n} \end{bmatrix}$$

Singular matrix

A square matrix that does not have a matrix inverse. A matrix is singular if its determinant is 0. For example, there are 10 2×2 singular $(0, 1)$-matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Symmetric and positive definite matrix

- A symmetric matrix is a square matrix that is equal to its transpose. Let A be a symmetric matrix. Then:

$$A = A^T.$$

  If the entries of matrix A are written as $A = (a_{ij})$, then the symmetric matrix A is such that $a_{ij} = a_{ji}$.

- An $n \times n$ real matrix M is positive definite if $z^T Mz > 0$ for all non-zero vectors z with real entries ($z \in \mathbb{R}^n$), where z^T denotes the transpose of z.

- An $n \times n$ complex matrix M is positive definite if $Re(z^* Mz) > 0$ for all non-zero complex vectors z, where z^* denotes the conjugate transpose of z and $Re(c)$ is the real part of a complex number c.

Examples

- The following matrix is symmetric:

$$\begin{bmatrix} 1 & 7 & 3 \\ 7 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}.$$

- Every diagonal matrix is symmetric, since all off-diagonal entries are zero.

Examples

- The nonnegative matrix

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is positive definite.

For a vector with entries

$$\mathbf{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$$

the quadratic form is

$$\begin{bmatrix} z_0 & z_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} z_0 \cdot 1 + z_1 \cdot 0 & z_0 \cdot 0 + z_1 \cdot 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = z_0^2 + z_1^2;$$

when the entries $z0, z1$ are real and at least one of them nonzero, this is positive.

A matrix in which some elements are negative may still be positive-definite. An example is given by

$$M_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

It is positive definite since for any non-zero vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

we have

$$
\begin{aligned}
x^{\mathrm{T}} M_1 x &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= \begin{bmatrix} (2x_1 - x_2) & (-x_1 + 2x_2 - x_3) & (-x_2 + 2x_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= 2{x_1}^2 - 2x_1x_2 + 2{x_2}^2 - 2x_2x_3 + 2{x_3}^2 \\
&= {x_1}^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + {x_3}^2
\end{aligned}
$$

which is a sum of squares and therefore nonnegative; in fact, each squared summa can be zero only when $x_1 = x_2 = x_3 = 0$, so M_1 is indeed positive-definite.

Conjugate transpose matrix

The conjugate transpose, Hermitian transpose, Hermitian conjugate, or adjoint matrix of an m-by-n matrix A with complex entries is the n-by-m matrix A^* obtained from A by taking the transpose and then taking the complex conjugate of each entry (i.e., negating their imaginary parts but not their real parts). The conjugate transpose is formally defined by

$$(\mathbf{A}^*)_{ij} = \overline{\mathbf{A}_{ji}}$$

where the subscripts denote the i, j-th entry, and the overbar denotes a scalar complex conjugate. (The complex conjugate of $a + bi$, where a and b are reals, is $a - bi$.)
This definition can also be written as

$$\mathbf{A}^* = (\overline{\mathbf{A}})^{\mathrm{T}} = \overline{\mathbf{A}^{\mathrm{T}}}$$

where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose and $\overline{\mathbf{A}}$, denotes the matrix with complex conjugated entries.

The conjugate transpose of a matrix A can be denoted by any of these symbols:

$$\mathbf{A}^* \text{ or } \mathbf{A}^{\mathrm{H}},$$

commonly used in linear algebra.
Example
If

$$\mathbf{A} = \begin{bmatrix} 3+i & 5 & -2i \\ 2-2i & i & -7-13i \end{bmatrix}$$

then

$$\mathbf{A}^* = \begin{bmatrix} 3-i & 2+2i \\ 5 & -i \\ 2i & -7+13i \end{bmatrix}$$

Basic remarks

- A square matrix A with entries a_{ij} is called Hermitian or self-adjoint if $A = A^*$, i.e., $a_{ij} = \overline{a_{ji}}$.
- normal if $A^*A = AA^*$.
- unitary if $A^* = A^{-1}$. a unitary matrix is a (square) $n \times n$ complex matrix A satisfying the condition $A^*A = AA^* = I_n$, where I_n is the identity matrix in n dimensions.
- Even if A is not square, the two matrices A^*A and AA^* are both Hermitian and in fact positive semi-definite matrices.
- Finding the conjugate transpose of a matrix A with real entries reduces to finding the transpose of A, as the conjugate of a real number is the number itself.

Rank

- Column rank of a matrix A is the maximum number of linearly independent column vectors of A. The row rank of a matrix A is the maximum number of linearly independent row vectors of A. Equivalently, the column rank of A is the dimension of the column space of A, while the row rank of A is the dimension of the row space of A.
- A result of fundamental importance in linear algebra is that the column rank and the row rank are always equal. It is commonly denoted by either $rk(A)$ or $rank\ A$. Since the column vectors of A are the row vectors of the transpose of A (denoted here by A^T), column rank of A equals row rank of A is equivalent to saying that the rank of a matrix is equal to the rank of its transpose, i.e. $rk(A) = rk(A^T)$.
- The rank of an $m \times n$ matrix cannot be greater than m nor n. A matrix that has a rank as large as possible is said to have full rank; otherwise, the matrix is rank deficient.

Cofactor

In linear algebra, the cofactor (sometimes called adjunct, see below) describes a particular construction that is useful for calculating both the determinant and inverse of square matrices. Specifically the cofactor of the (i, j) entry of a matrix, also known as the (i, j) cofactor of that matrix, is the signed minor of that entry.

Informal approach to minors and cofactors

Finding the minors of a matrix A is a multi-step process:

- Choose an entry a_{ij} from the matrix.
- Cross out the entries that lie in the corresponding row i and column j.
- Rewrite the matrix without the marked entries.
- Obtain the determinant M_{ij} of this new matrix.

If $i + j$ is an even number, the cofactor C_{ij} of a_{ij} coincides with its minor: $C_{ij} = M_{ij}$.
Otherwise, it is equal to the additive inverse of its minor: $C_{ij} = -M_{ij}$.

Formal definition of cofactor

If A is a square matrix, then the minor of its entry a_{ij}, also known as the (i, j) minor of A, is denoted by M_{ij} and is defined to be the determinant of the submatrix obtained by removing from A its i-th row and j-th column.
It follows: $C_{ij} = (-1)^{i+j} M_{ij}$ and C_{ij} is called the cofactor of a_{ij}.

Example

Given the matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

suppose we wish to find the cofactor C_{23}. The minor M_{23} is the determinant of the above matrix with row 2 and column 3 removed.

$$M_{23} = \begin{vmatrix} b_{11} & b_{12} & \square \\ \square & \square & \square \\ b_{31} & b_{32} & \square \end{vmatrix} \text{ yields } M_{23} = \begin{vmatrix} b_{11} & b_{12} \\ b_{31} & b_{32} \end{vmatrix} = b_{11} b_{32} - b_{31} b_{12}$$

Using the given definition it follows that

$$C_{23} = (-1)^{2+3}(M_{23})$$
$$C_{23} = (-1)^{5}(b_{11} b_{32} - b_{31} b_{12})$$
$$C_{23} = b_{31} b_{12} - b_{11} b_{32}.$$

Invertible matrix

- In linear algebra an n-by-n (square) matrix A is called invertible (some authors use nonsingular or nondegenerate) if there exists an n-by-n matrix B such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, where I_n denotes the n-by-n identity matrix and the multiplication used is ordinary matrix multiplication. If this is the case, then the matrix B is uniquely determined by A and is called the inverse of A, denoted by A^{-1}. It follows from the theory of matrices that if $\mathbf{AB} = \mathbf{I}$ for finite square matrices A and B, then also $\mathbf{BA} = \mathbf{I}$.
- Non-square matrices (m-by-n matrices which do not have an inverse). However, in some cases such a matrix may have a left inverse or right inverse. If A is m-by-n and the rank of A is equal to n, then A has a left inverse: an n-by-m matrix B such that $BA = I$. If A has rank m, then it has a right inverse: an n-by-m matrix B such that $AB = I$.
- A square matrix that is not invertible is called singular or degenerate. A square matrix is singular if and only if its determinant is 0.

How to find inverse matrix

- Analytic solution

  Writing the transpose of the matrix of cofactors, known as an adjugate matrix, can also be an efficient way to calculate the inverse of small matrices, but this recursive method is inefficient for large matrices. To determine the inverse, we calculate a matrix of cofactors:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\left(\mathbf{C}^{\mathrm{T}}\right)_{ij} = \frac{1}{|\mathbf{A}|}\left(\mathbf{C}_{ji}\right) = \frac{1}{|\mathbf{A}|}\begin{pmatrix} \mathbf{C}_{11} & \mathbf{C}_{21} & \cdots & \mathbf{C}_{n1} \\ \mathbf{C}_{12} & \mathbf{C}_{22} & \cdots & \mathbf{C}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_{1n} & \mathbf{C}_{2n} & \cdots & \mathbf{C}_{nn} \end{pmatrix}$$

  where $|A|$ is the determinant of A, C_{ij} is the matrix of cofactors, and C^T represents the matrix transpose.

Inversion of 2×2 matrices

The cofactor equation listed above yields the following result for 2×2 matrices. Inversion of these matrices can be done easily as follows:

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This is possible because $1/(ad - bc)$ is the reciprocal of the determinant of the matrix in question, and the same strategy could be used for other matrix sizes.

Inversion of 3×3 matrices

A computationally efficient 3×3 matrix inversion is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix}^{T} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & K \end{bmatrix}$$

where the determinant of A can be computed by applying the rule of Sarrus as follows:
$\det(\mathbf{A}) = a(ek - fh) - b(kd - fg) + c(dh - eg)$.
If the determinant is non-zero, the matrix is invertible, with the elements of the above matrix on the right side given by

$$\begin{array}{lll} A = (ek - fh) & D = (ch - bk) & G = (bf - ce) \\ B = (fg - dk) & E = (ak - cg) & H = (cd - af) \\ C = (dh - eg) & F = (gb - ah) & K = (ae - bd). \end{array}$$