Numerisk Analys, MMG410. Lecture 5.

Vektornormer

Example

$$x = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}; \qquad \begin{aligned} ||x||_1 &= |-1| + |2| + |3| + |-5| = 11 \\ ||x||_2 &= \sqrt{(-1)^2 + 2^2 + 3^2 + (-5)^2} = \sqrt{39} \end{aligned}$$

$||x||_\infty = max(|-1|, |2|, |3|, |-5|) = 5.$

En vektor x är normerad om $||x|| = 1$. $x \neq 0 \implies \dfrac{x}{||x||}$

$$x^T x = [-1,\ 2,\ 3,\ -5] \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix} = (-1)\cdot(-1) + 2\cdot 2 + 3\cdot 3 + (-5)^2 = 39$$

$||x||_2 = \sqrt{39}$

$$V = \frac{x}{||x||} = \left[\frac{-1}{\sqrt{39}}, \frac{2}{\sqrt{39}}, \frac{3}{\sqrt{39}}, \frac{-5}{\sqrt{39}}\right]^T \implies ||V||_2 = 1$$

Matrisnormer

Example

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\begin{aligned} ||A||_1 &= \max_k \sum_{r=1}^{m} |a_r, k| \\ ||A||_2 &= \max \sqrt{\lambda(A^T A)} \\ ||A||_\infty &= \max_r \sum_{k=1}^{n} |a_r, k| \end{aligned}$$

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 6 & 4 & 2 \\ 9 & -6 & 3 \end{bmatrix}, \quad \begin{aligned} ||A||_1 &= \max(|1| + |6| + |9|, |-2| + |4| + \\ &+ |-6|, |-3| + |2| + |3|) = \\ &= \max(16, 12, 8) = 16 \end{aligned}$$

$$\begin{aligned} ||A||_\infty &= \max(|1| + |-2| + |-3|, |6| + |4| + |2|, \\ &|9| + |-6| + |3|) = \max(6, 12, 18) = 18 \end{aligned}$$

Matrisnormer

Example

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 6 & 4 & 2 \\ 9 & -6 & 3 \end{bmatrix}$$

$||A||_2 = \max \sqrt{\lambda(A^T A)}$

$$A^T = \begin{bmatrix} 1 & 6 & 9 \\ -2 & 4 & -6 \\ -3 & 2 & 3 \end{bmatrix}, \; A^T A = \begin{bmatrix} 118 & -32 & 36 \\ -32 & 56 & -4 \\ 36 & -4 & 22 \end{bmatrix}$$

$$\lambda(A^T A) = \begin{bmatrix} 8.9683 \\ 45.3229 \\ 141.7089 \end{bmatrix}; \quad \max \sqrt{\lambda(A^T A)} = \max(2.9947, 6.7322, 11.9042) = 11.9042$$

Matrisnormer

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \ A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \ A^T A - \lambda I = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0;$$

$$\lambda_1 = 1, \ \lambda_2 = 1; \ ||A||_2 = \max \sqrt{\lambda(A^T A)} = \max(1,1) = 1$$