Leading submatrix

$$A = \underbrace{\begin{bmatrix} \boxed{a_{11}} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n}$$

$= A(1:1;\,1:1)$

$A(1:j;\,1:j)$ - leading submatrix av A

Om $\det(A(1:j;\,1:j)) \neq 0$, då existerar $A = LU$.

Exempel:

$$A = \begin{bmatrix} 2 & 6 \\ 4 & 15 \end{bmatrix}$$

Kollar: $\det(A(1:1;\,1:1)) = 2 > 0$

$\det(A(1:2;\,1:2)) = \det\left(\begin{bmatrix} 2 & 6 \\ 4 & 15 \end{bmatrix}\right) =$

$= 2 \cdot 15 - 24 = 6 > 0$

Vi kan göra $A = LU$ faktorizing

$$A = \begin{bmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$A_1 \quad A_2$

$\det A_1 \neq 0$

$\det A_2 \neq 0$

$$\boxed{A = L \cdot U}$$

$\rightarrow$

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

$$A_n = \begin{array}{c|cc} & j & \\ \hline j & \boxed{A_{11}} & A_{12} \\ & A_{21} & A_{22} \end{array} =$$

$$= L \cdot U = \begin{bmatrix} L_{11} & \cancel{0} \\ L_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} L_{11} \cdot U_{11}; & L_{11} \cdot U_{12} \\ L_{21} \cdot U_{11}; & L_{21} \cdot U_{12} + L_{22} \cdot U_{22} \end{bmatrix}$$

$$\det(A_{11}) = \det(L_{11} \cdot U_{11})$$
$$= \det(L_{11}) \cdot \det(U_{11}) = 1 \cdot (\neq 0$$

Ex. 1/1

$$A = \begin{bmatrix} 2 ; & 6 \\ 4 ; & 15 \end{bmatrix}$$

$$A = L \cdot U$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$$

$$\cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} l_{11} \cdot u_{11} ; & l_{11} \cdot u_{12} \\ l_{21} \cdot u_{11} ; & l_{21} \cdot u_{12} + l_{22} u_{22} \end{bmatrix}$$

$$a_{11} = l_{11} \cdot u_{11}$$

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$$

EX. 1/2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} \cdot u_{11} ; & l_{11} \cdot u_{12} \\ l_{21} \cdot u_{11} ; & l_{21} \cdot u_{12} + l_{22} \cdot u_{22} \end{bmatrix}$$

$$\underline{a_{11}} = l_{11} \cdot u_{11} = 1 \cdot u_{11}$$

$$a_{12} = l_{11} \cdot u_{12} = 1 \cdot u_{12}$$

$$a_{21} = l_{21} \cdot u_{11} \rightarrow$$

$$\rightarrow l_{21} = \frac{a_{21}}{u_{11}} = \frac{a_{21}}{a_{11}}$$

$$L = \begin{bmatrix} 1 & 0 \\ \frac{a_{21}}{a_{11}} & 1 \end{bmatrix} ; U = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \bullet \end{pmatrix}$$

ex. 1/3

$$a_{22} = l_{21} \cdot u_{12} + l_{22} \cdot u_{22}$$

$$l_{22} u_{22} = a_{22} - l_{21} \cdot u_{12}$$

$$u_{22} = \frac{a_{22} - l_{21} \cdot u_{12}}{l_{22}}$$

$$l_{22} = 1$$

$$L = \begin{pmatrix} 1 & 0 \\ \underbrace{\frac{a_{21}}{a_{11}}}_{=2} & 1 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 6 \\ 4 & 15 \end{bmatrix} \quad l_{21} = \frac{a_{21}}{a_{11}} = \frac{4}{2} = 2$$

$$= A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$$

Kap. 2, Övn. 10

Beräkna LU faktoriseringen av matrisen nedan: $A = \begin{bmatrix} 1 & a \\ c & b \end{bmatrix}$ När är matrisen singulär?

Svar

$$A = \begin{bmatrix} 1 & a \\ c & b \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}}_{U} =$$

$$= LU = \begin{bmatrix} u_{11} ; & u_{12} \\ l_{21} \cdot u_{11} ; & l_{21} \cdot u_{12} + u_{22} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$$

$$u_{11} = a_{11} = 1$$

$$u_{12} = a_{12} = a$$

$$l_{21} \cdot u_{11} = c \Rightarrow l_{21} = \frac{c}{u_{11}} = \frac{c}{1} = c$$

$$l_{21} \cdot u_{12} + u_{22} = b$$

$$u_{22} = b - l_{21} \cdot u_{12} = b - c \cdot a$$

LU faktoriseringen:

$$\begin{bmatrix} 1 & a \\ c & b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & a \\ 0 & b-ca \end{bmatrix}}_{U}$$

A är singulär när $b = ca$, då $U = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$ och $\det U = 0 \Rightarrow \det A = 0$.

Cholesky fakt.

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$$

$$A = L \cdot L^T$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}}_{L} \cdot \underbrace{L^T}_{=\begin{bmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{bmatrix}}$$

$$f(x,y) = \sin(x^2) + \cos(y^2)$$

$$H = ? \; ; \; (x,y) = (0,0)$$

$$A = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

$$A^{-1} = \frac{1}{\det A} \cdot \bar{C}^{T} = \frac{1}{ad-cb} \left[\bar{C}^{T} \right]$$

$$\bar{C} = \begin{bmatrix} (-1)^{1+1} \cdot d \; ; & (-1)^{1+2} \cdot c \\ (-1)^{1+2} \cdot b \; ; & (-1)^{2+2} \cdot a \end{bmatrix} =$$

$$= \begin{bmatrix} d \; ; & -c \\ -b \; ; & a \end{bmatrix} ; \quad \bar{C}^{T} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inversion of 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} := \frac{1}{\det A} \cdot C^T$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

C for A?

$$C_{11} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{1+1} \cdot d = d$$

$$C_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{1+2} \cdot c = -c$$

$$C_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{2+1} \cdot b = -b$$

$$C_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (-1)^{2+2} \cdot a = a$$

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} d ; & -b \\ -c ; & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot \begin{bmatrix} d ; & -b \\ -c ; & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d ; & -b \\ -c ; & a \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C = \begin{bmatrix} ek-fh; & gf-dk; & dh-ge \\ hc-bk; & ak-gc; & gb-ah \\ bf-ec; & dc-af; & ae-db \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \cdot C^T$$

$$C_{11} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} e & f \\ h & k \end{vmatrix} = ek - fh$$

$$C_{12} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{12} = (-1)^{1+2} \begin{vmatrix} d & f \\ g & k \end{vmatrix} = -(dk - gf)$$

$$C_{13} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{13} = (-1)^{1+3} \begin{vmatrix} d & e \\ g & h \end{vmatrix} = dh - ge$$

$$C_{21} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{21} = (-1)^{2+1} \begin{vmatrix} b & c \\ h & k \end{vmatrix} = -(bk - hc)$$

$$C_{22} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{22} = (-1)^{2+2} \begin{vmatrix} a & c \\ g & k \end{vmatrix} = ak - gc$$

$$C_{23} \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{23} = (-1)^{2+3} \begin{vmatrix} a & b \\ g & h \end{vmatrix} = -(ah - gb)$$

$$C_{31} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{31} = (-1)^{3+1} \cdot \begin{vmatrix} b & c \\ e & f \end{vmatrix} = bf - ec$$

$$C_{32} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{32} = (-1)^{3+2} \cdot \begin{vmatrix} a & c \\ d & f \end{vmatrix} = -(af - dc)$$

$$C_{33} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}; \quad C_{33} = (-1)^{3+3} \cdot \begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae - db$$