Fixpunkter och Newtons metod

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)} \quad \text{Newtons m.}$$

$$x^{k+1} = g(x^k) \quad \text{Fixpunktsiteration}$$

$$\boxed{g(x) = x - \frac{f(x)}{f'(x)}} \Rightarrow \text{för fixpunkt } x^*$$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x) \cdot f''(x)}{(f'(x))^2}$$

$$g'(x^*) = 1 - \frac{(f'(x^*))^2 - \overbrace{f(x^*)}^{=0} \cdot f''(x^*)}{(f'(x^*))^2} = 0$$

Vi visar kvadratisk konvergens för enkelrot x^*. Inför $\boxed{\delta_k := x_k - x^*}$

$$\underbrace{x_{k+1} - x^*}_{\delta_{k+1}} = \underbrace{x_k - x^*}_{\delta_k} - \frac{f(x_k)}{f'(x_k)} =$$

$$= x_k - x^* - \frac{f(x^* + x_k - x^*)}{f'(x^* + x_k - x^*)};$$

$$\delta_{k+1} = \delta_k - \frac{f(x^* + \delta_k)}{f'(x^* + \delta_k)} \rightarrow \text{använder Taylor's formula}$$

$$\sigma_{k+1} = \sigma_k - \frac{\underset{=0}{f(x^*)} + \sigma_k f'(x^*) + \sigma_k^2 f''(x^*)/2 + \dots}{f'(x^*) + \sigma_k f''(x^*) + \dots}$$

$$= \frac{\sigma_k \cdot \left(f'(x^*) + \sigma_k f''(x^*) + \dots\right) - \underset{=0}{f(x^*)} - \sigma_k f'(x^*) - \frac{\sigma_k^2 f''(x^*)}{2} - \dots}{f'(x^*) + \sigma_k f''(x^*) + \dots}$$

$$= \frac{\frac{\sigma_k^2 f''(x^*)}{2} + \dots}{\left(f'(x^*) + \sigma_k f''(x^*) + \dots\right)} \approx \frac{f''(x^*)}{2 f'(x^*)} \sigma_k^2$$

Vi har fått kvadratisk konvergens

$$\underbrace{\sigma_{k+1}}_{x_{k+1} - x^*} \approx \frac{f''(x^*)}{2 f'(x^*)} \sigma_k^2$$

" $(x_k - x^*)^2$

$$\frac{|\sigma_{k+1}|}{|\sigma_k^{2}|} \approx \frac{f''(x^*)}{2 f'(x^*)}$$

konv. ordning.

" Const < 1 för konvergens

Exempel 1 (fixpunkter)

$$g(x) = x^2$$

Fixpunktsiteration:

$$\boxed{x^{k+1} = g(x^k)}$$

eller

$$\boxed{x^{k+1} = (x^k)^2}$$

Fixpunkter: $x^* = g(x^*) \Rightarrow$

$$x^* = (x^*)^2 \Rightarrow x^* - (x^*)^2 = 0$$

$$x^*(1 - x^*) = 0 \Rightarrow$$

Fixpunkter: $\boxed{x^* = 0, \quad x^* = 1}$

fixpunkter

Konvergens:

$$\boxed{|g'(x^*)| < 1}$$

← för konvergensen av fixpunkter.

$g'(x) = 2x$; $x^* = 0 \rightarrow |g'(0)| = 0 < 1$

$\Rightarrow$ konvergerar med $x^{(0)} = 0$.

$g'(x) = 2x$; $x^* = 1 \rightarrow |g'(1)| = |2 \cdot 1| > 1$

— divergens för $x^0 = 1$.

Bra startvärde: $x^{(0)} = 0$

Exempel 2.

$$g(x) = \frac{x}{2}$$

Fixpunktsiteration:

$$x^{k+1} = g(x^k) = \frac{x^k}{2}$$

Konvergens och fixpunkter:

$$x^* = g(x^*) \rightarrow x^* - \frac{x^*}{2} = 0$$

$$\rightarrow \frac{2x^* - x^*}{2} = 0 \rightarrow \boxed{x^* = 0}$$

✓ fixpunkt

Konvergens:

$$|g'(x)| = \left|\frac{1}{2}\right| = \text{const.} < 1 \quad \text{alltid!!!}$$

konvergerar → ej jämt.

Exempel 3.

$$g(x) = \cos x$$

Fixpunktsiteration:

$$x^{k+1} = g(x^k) = \cos(x^k)$$

Fixpunkter: $x^* = g(x^*) \rightarrow x^* = \cos x^*$

$$\rightarrow x^* - \cos x^* = 0 \rightarrow \boxed{x^* \approx 0.739}$$

Fixpunkt

Konvergens: Ja! Vi har konvergens:

$$|g'(x)| = |-\sin x| < 1; \text{ FÖR } x^* = 0.739$$

$$|g'(0.739)| = |-\sin(0.739)| \approx 0.674 < 1$$

K.3/1

Exempel 3:

Lös $\frac{x^2-2}{\alpha} = 0$ med Newton's och fixpunktsiteration metoden

Svar:

$$f(x) = \frac{x^2-2}{\alpha}; \quad f'(x) = \frac{2x\cdot\alpha}{\alpha^2} = \frac{2x}{\alpha}$$

Newton's m:

$$\boxed{x_{k+1} = x_k - \frac{x_k^2-2}{\alpha} \Big/ \frac{2x_k}{\alpha}} =$$

$$= \underbrace{x_k - \frac{x_k^2-2}{2x_k}}_{g(x_k)}$$

beror inte av α

$$g(x) = x - \frac{x^2-2}{2x} = \frac{2x^2-x^2+2}{2x} =$$

$$= \frac{2+x^2}{2x};$$

Fixpunktsiteration:

$$\boxed{x^{k+1} = g(x^k) = \frac{2+x_k^2}{2x_k}}$$

Fixpunkter:

$$x^* = g(x^*) \to x^* - \frac{2+x_*^2}{2x^*} = 0$$

$$\frac{2x^{*2}-2-x^{*2}}{2x^*} = 0 \to \frac{(x^*)^2-2}{2x^*} = 0$$

$$\boxed{\text{Fixpunkter: } x^* = \pm\sqrt{2}}$$

Ex3/2

Konvergens:

$$g(x) = \frac{2+x^2}{2x}$$

$$g'(x) = \frac{(2+x^2)' \cdot 2x - (2+x^2)(2x)'}{4x^2} =$$

$$= \frac{(2x)^2 - 4 - 2x^2}{4x^2} = \frac{2x^2-4}{4x^2} = \frac{x^2-2}{2x^2}$$

$|g'(x^*)| < 1 \rightarrow$ för konvergensen.

1) $x^* = \sqrt{2}$: $|g'(x^*)| = \left|\frac{(\sqrt{2})^2-2}{2(\sqrt{2})^2}\right| = 0 < 1$ $\rightarrow$ konvergens

2) $x^* = -\sqrt{2} \rightarrow |g'(x^*)| = 0 < 1$ – konvergens.

Newtons: $\boxed{x^{k+1} = x^k - \left(\frac{x_k^2-2}{\alpha} \bigg/ \frac{2x_k}{\alpha}\right)}$

skriver om $f(x)$

Med $\boxed{0 = f(x) = \frac{x^2-2}{\alpha} + x = x}$ $f(x) = \frac{x^2-2}{\alpha}$

$g(x)$

Om $\boxed{g(x) = \frac{x^2-2}{\alpha} + x}$, $\boxed{g'(x) = \frac{2x}{\alpha} + 1}$

Fixpunkter: $\boxed{\pm\sqrt{2}}$; $g'(\sqrt{2}) = \left|\frac{2\cdot\sqrt{2}}{\alpha} + 1\right| < 1$

$\boxed{g'(-\sqrt{2}) = \left|-\frac{2\sqrt{2}}{\alpha} + 1\right| < 1.}$ Låt $\alpha = 3$.

$g'(\sqrt{2}) = \left|-\frac{2\sqrt{2}}{3} + 1\right| \approx 0.05719 < 1$

snabb konvergens

Ex.4/1

Exempel 4.

Lös $f(x) = x^2 - 2 = 0$

Newton's metod:

$$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$$

$$x^{k+1} = x^k - \frac{(x^k)^2 - 2}{2x^k} = g(x^k)$$

Vi kan också använd fixpunktiter. metod och skriver

$$g(x) = \cancel{x} \frac{x^2-2}{2x} = x - \frac{x^2-2}{2x} =$$

$$= \frac{2x^2 - x^2 + 2}{2x} = \frac{x^2+2}{2x}$$

Fixpunkter:

$$x^* = \frac{(x^*)^2+2}{2x^*} \rightarrow x^* - \frac{(x^*)^2+2}{2x^*} = 0$$

$$\frac{2(x^*)^2 - (x^*)^2 - 2}{2x^*} = 0 \rightarrow \frac{(x^*)^2-2}{2x^*} = 0$$

$$x^* = \pm\sqrt{2} \text{ - fixpunkter}$$

Konvergens:

$$g'(x) = \frac{(2x)'(x^2+2) - (x^2+2)'2x}{4x^2} =$$

$$= (2(x^2+2) - (2x)^2) / 4x^2 =$$

Ex. 4/2)

$$= \frac{2x^2+4-4x^2}{4x^2} = \frac{4-2x^2}{4x^2}$$

$$g'(\pm\sqrt{2}) = \frac{4-2\cdot(\pm\sqrt{2})^2}{4(\pm\sqrt{2})^2} = \frac{4-2\cdot 2}{4\cdot 2} = 0$$

$$\boxed{|g'(\pm\sqrt{2})| = 0 < 1 \text{ - konvergens}}$$

För fixpunktiteration:

$$x^{k+1} = g(x^k) = \frac{x_k^2+2}{2x_k}$$

Bra startvärde: $x_0 = 1$